LIFE INSURANCE MATHEMATICS

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PREFACE

This book is intended as a textbook for courses in life insurance mathematics in colleges and universities. It is hoped that the book will also prove valuable to persons, other than college students, who are interested in life insurance. In particular, we hope that prospective actuaries will profit from reading this book before attempting the more advanced works in life contingencies recommended by the Society of Actuaries. This book is not, however, aimed primarily at the actuarial student. Rather, it is aimed at the college student who has more than a superficial interest in life insurance and who has a reasonable amount of mathematical aptitude.

For the study of this volume no mathematical preparation beyond the high-school level is required. It is suggested, however, that the prerequisite for a course based on this text be a course in mathematics of finance, but that the prerequisite be waived for anyone with a reasonable amount of mathematical experience, such as three courses in college mathematics. On the face of it, this recommendation is inconsistent with the statement that no mathematical preparation beyond the high-school level is required for a study of this book. The reason for the apparent inconsistency is our belief that few students really understand high-school algebra without some experience with mathematics at the college level.

We feel that the subject of life insurance mathematics deserves more recognition in college curricula than it is currently receiving. Our feeling in this respect is based on the following major reasons:

1. The institution of life insurance is extremely important in American economy. Furthermore, almost everyone is directly affected by it sooner or later. Some knowledge of the mathematical principles is essential to a thorough understanding of the subject.

2. There is a regrettable tendency for many students to avoid the study of mathematics, and its valuable mental training, on the grounds that pure mathematics will not be useful to them. Our experience at the University of Wisconsin indicates that a course in life insurance mathematics will attract a number of such students, thus exposing them to mathematical training which they would otherwise miss.

3. With the growth of pension plans and the increasing complexities of the life insurance business, more actuaries are needed. Many students who would be happy and successful in actuarial work go through their college careers without fully realizing that the profession of actuary exists.

The major reason for publishing a book on life insurance mathematics at the present time is that existing books are out of date on the following counts:

1. Shortly after World War II the Executive Council of the Permanent Committee, International Congress of Actuaries, approved a new actuarial notation. The new notation is not used in existing books.

2. The examples in existing books are based on old mortality tables and on rates of interest that are no longer realistic in view of the spectacular decline in interest rates over the last 20 years.

3. Adoption by the several states of the Standard Valuation and Non-Forfeiture Laws necessitates a new treatment for the subjects of valuation and non-forfeiture benefits.

This book is, of course, up to date on these points.

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1 • INTRODUCTION

Every plan of insurance is, in effect, a cooperative arrangement whereby the members of an association or the policyholders of an insurance company band together to share a financial loss which may be too serious to be borne by the individual affected. Life insurance is primarily concerned with the financial loss incident to death. Death produces a financial loss because of (*a*) the expense incidental to death, which includes such things as the expense of last illness, the undertaker's fee, and inheritance tax, and (*b*) the loss of earning power of the deceased individual. The real reason for the existence of life insurance is compounded from the financial loss incident to death and from the fact that it is impossible to predict when an individual in reasonably good health will die. Although we can't predict the time of death of a particular individual, we can be certain that everyone will die sooner or later. Moreover, if we make the reasonable assumption that conditions affecting deaths in the immediate future will not differ radically from conditions in the immediate past, we can make general predictions regarding future deaths from studies made of past experience. We can predict, for example, that out of a large group of persons chosen at random approximately so many will die in a given period, even though we cannot say which particular individuals will die.

Fundamentally the structure of life insurance depends upon three elements: (*a*) mortality, i.e., the probability of the death of an individual in a given period of time; (*b*) interest, i.e., the rate of interest that can be earned on invested funds; and (*c*) expense, i.e., the expenses incident to the sale and maintenance of a life insurance policy. In the first part of this book our attention will be confined to mortality and interest; we shall assume that there are no expenses incurred in connection with the handling of a life insurance policy. The problem of expenses will be discussed in Chapters 7, 8, and 9.

In order to understand the role of mortality in life insurance calculations it is necessary to acquire an understanding of the elements of the theory of probability. Accordingly, the next four sections of this chapter will be concerned with the basic principles of the theory of probability.

2 · DEFINITIONS OF PROBABILITY

(*a*) THE "A PRIORI" DEFINITION. If an event can happen in h ways and fail in f ways, and if each of these ways is equally likely, then the probability that the event will happen is given by

$$p = \frac{h}{h + f}$$

and the probability that the event will fail to happen is given by

$$q = \frac{f}{h + f}.$$

If, for example, we draw 1 card from an ordinary deck of 52, the probability of drawing an ace is $4/52 = 1/13$ because there are 4 ways of drawing an ace ($h = 4$) and 48 ways of drawing a card that is not an ace ($f = 48$); moreover, each of these 52 ways is equally likely. We are just as likely to draw the 5 of spades, for instance, as any other card.

Let's consider another example: What is the probability of getting a 7 in a throw of 2 ordinary dice? It is a temptation to say that the probability is $1/11$ because a throw of 2 dice may produce 11 different results (2 to 12, inclusive) and only one of these is a 7. This reasoning sounds plausible on the face of it, and yet the result does not sound plausible. Both the reasoning and the result are, in fact, incorrect. The fault lies in the fact that the 11 ways are not equally likely. Most of us know intuitively that a 7 is more likely than, for instance, a deuce. Closer reasoning will throw light on this problem.

Suppose that one of the dice is red and the other white. This assumption will clarify the problem without affecting the generality of our conclusion. There are 6 ways of throwing the red die. All of these ways are equally likely. For every one of these 6 ways there are 6 equally likely ways of throwing the white die—or 36 equally likely ways of throwing the 2 dice together. The accompanying table shows in detail what these 36 ways are.

Red Die	*White Die*	*Total*
1	1	2
1	2	3
1	3	4
1	4	5
1	5	6
1	6	7
2	1	3
2	2	4
2	3	5
2	4	6
2	5	7
2	6	8
3	1	4
3	2	5
3	3	6
3	4	7
3	5	8
3	6	9
4	1	5
4	2	6
4	3	7
4	4	8
4	5	9
4	6	10
5	1	6
5	2	7
5	3	8
5	4	9
5	5	10
5	6	11
6	1	7
6	2	8
6	3	9
6	4	10
6	5	11
6	6	12

From the table above we can readily form the table below:

Total	*Number of Ways*	*Total*	*Number of Ways*
2	1	8	5
3	2	9	4
4	3	10	3
5	4	11	2
6	5	12	1
7	6		36

Accordingly, h in our problem is 6, and $h + f$ is 36, so that the probability of getting a 7 is $\frac{6}{36} = \frac{1}{6}$. Similarly, the probability of getting a 4 is $\frac{3}{36} = \frac{1}{12}$. And so forth.

(*b*) Empirical probability. In many problems it is impossible to analyze an event into the number of equally likely ways it can happen or fail. A notable example is the field of life insurance. In such problems it is frequently possible to observe a number of trials of the event and to record the number of happenings. The best estimate we can make of the probability of the event is the ratio of the number of happenings to the number of trials. The larger the number of trials, the better this estimate will be. If we assume that experience in the future will be consistent with experience in the past, we can define the probability of the occurrences of an event, in mathematical language, as

$$p = \lim_{n\to\infty} \frac{m}{n},^*$$

where m is the number of happenings and n the number of trials. The number of failures is, of course, $n - m$.

Suppose, for example, that the records of a life insurance company—or of several life insurance companies which pool their data—show

* $\lim\limits_{n\to\infty} \frac{m}{n}$ is read "the limit of m over n as n becomes infinitely large."

The reader with limited mathematical experience may acquire some insight into the concept of "limit" by considering the following example.

Let
$$S_1 = \frac{1}{2},$$
$$S_2 = \frac{1}{2} + \left(\frac{1}{2}\right)^2 = \frac{3}{4},$$
$$S_3 = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 = \frac{7}{8},$$
$$S_4 = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 = \frac{15}{16},$$
$$\cdots$$
$$S_n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots + \left(\frac{1}{2}\right)^n = \frac{2^n - 1}{2^n}.$$

It is apparent that, as n increases, S_n gets closer and closer to 1. We say that in the limit, as n becomes infinite, $S_n = 1$. Or, in mathematical notation, $\lim\limits_{n\to\infty} S_n = 1$.

that, out of 100,000 men alive at age 35, 99,700 lived to age 36. A reasonable estimate of the probability that an insured man aged 35 will live at least 1 year would be 99,700/100,000 or 0.997.

In arriving at the figure of 0.997 it is tacitly assumed that we know nothing about how the man is classified except that he is 35 years old and insured.

Suppose that a more detailed analysis of the company records show that 40 of the 300 deaths occurred among the 10,000 men who were classified as substandard lives because of some hazard connected with their occupation; 60 deaths occurred among the 10,000 men who were classified as substandard lives because of a medical impairment; and the remaining 200 deaths occurred among the 80,000 men who were classified as standard lives. Then the best available estimate of the probability that an insured man aged 35 will live at least 1 year would be:

(*a*) $\frac{79{,}800}{80{,}000}$, or 0.9975, if the man is known to be classified as a standard risk.

(*b*) $\frac{9{,}960}{10{,}000}$, or 0.9960, if the man is known to be classified as substandard due to occupation.

(*c*) $\frac{9{,}940}{10{,}000}$, or 0.9940, if the man is known to be classified as substandard due to medical impairment.

(*d*) $\frac{99{,}700}{100{,}000}$, or 0.9970, if nothing is known about the man's classification.

In theory, these classifications could be subdivided indefinitely. As a practical matter, however, the empirical probabilities become less reliable as the number of lives in a classification becomes smaller. For this reason life insurance companies classify risks in broad groups. A second reason is that a great many classifications would involve too heavy administrative expenses.

3 · CONCLUSIONS TO BE DRAWN FROM THE DEFINITIONS OF PROBABILITY

(*a*) The probability of an event can never be less than 0 nor more than 1. If an event is certain to happen, the probability of its happening is 1; if it is certain to fail, the probability of its happening is 0.

(*b*) The sum of the probabilities of the happening and of the failing of an event is 1. That is, $p + q = 1$.

(*c*) If the probability of the happening of an event is known, the probability of its failing is also known.

4 · MATHEMATICAL EXPECTATION

If p is the probability that a person will be successful in any venture and M is the amount of money he will receive if successful, his expectation will be represented by an amount of money equal to pM.

Suppose, for example, that a man is to receive \$3 if he throws a 7 in a single throw of 2 dice. His expectation is equal to the product of (*a*) the probability of throwing a 7 and (*b*) \$3. That is, $\frac{1}{6} \times 3$, or 50 cents.

Consider a somewhat more complicated example. Suppose that a man is to receive \$3 if he throws a 7 and \$9 if he throws a 12 in a single throw with 2 dice. His expectation is equal to $(\frac{1}{6} \times 3) + (\frac{1}{36} \times 9)$, or 75 cents. This means that 75 cents would be a fair price to pay for the privilege of throwing the dice under the given conditions. It also means that a man with unlimited funds, no overhead expense, no desire for profit, and no objections to pure gambling if done by other persons could charge 75 cents for the privilege of throwing the dice for the rewards mentioned. He could expect to break even in the long run if enough people played his game.

The concept of expectation is not confined to reference to a person. It may refer to things, in which case the term probable value is frequently used. For example, the expected number of 7's in 600 throws with 2 dice is $\frac{1}{6} \times 600$, or 100. This does not mean that if we throw 2 dice 600 times we are sure to produce 7 100 times. It simply means that 100 is the most probable value for the number of times 7 will turn up.

5 · THEOREMS ON PROBABILITY

Two or more events are said to be mutually exclusive if not more than one of them can happen in a single trial. Thus, in drawing 1 card from a deck of 52 the drawing of an ace and the drawing of a king are mutually exclusive. But the drawing of an ace and the drawing of a spade are not.

Two events are said to be independent if the happening of either one of ~~them has no~~ effect on the happening of the other. Two events are said to be dependent if the happening of one affects the happening of the other. Thus, throwing a 5 with the red die and throwing a 5 with the white die are independent events. But the probability of throwing

a 5 with the red die and the probability of throwing a 7 with both dice are dependent events.

THEOREM I. If $p_1, p_2, p_3, \cdots, p_n$ are the probabilities of the occurrence of n mutually exclusive events, then the probability that one of these events will happen is

$$p_1 + p_2 + p_3 + \cdots + p_n.$$

THEOREM II. If $p_1, p_2, p_3, \cdots, p_n$ are the probabilities of the occurrence of n independent events, then the probability of the occurrence of all of these events is

$$p_1 \cdot p_2 \cdot p_3 \cdots p_n.$$

THEOREM III. If the probability of the occurrence of a first event is p_1, and if after it bas happened the probability of the occurrence of a second event is p_2, tbe probability that both events will happen in the specified order is $p_1 \cdot p_2$.

To many readers these theorems will appear intuitively obvious on reflection. Formal mathematical proofs can be found in any standard textbook covering the elementary theory of probability.

EXAMPLE 1

What is the probability of not getting a 7 in a throw of 2 ordinary dice?

Solution: Throwing a 7 and not throwing a 7 are mutually exclusive events. Furthermore, one of the two events is certain to happen. Therefore, the sum of tbe probabilities of the two events is unity. Since the probability of throwing a 7 is $1/6$, according to section 2, the probability of not throwing a 7 is $1 - 1/6 = 5/6$.

EXAMPLE 2

If the probabilities that Jones and Smith survive a certain period are 0.8 and 0.9, respectively, what is the probability that (*a*) both Jones and Smith survive? (*b*) at least one dies?

Solution: (*a*) Assuming that the deaths of Jones and Smith are independent events, theorem II applies, and we have $0.8 \times 0.9 = 0.72$ as the required probability.

(*b*) Either they both survive, or at least one dies. Moreover, these two events are mutually exclusive. Hence the sum of the two respective probabilities is unity, and we have $1 - 0.72 = 0.28$ as the required probability.

Part *b* can be solved by another approach which, although longer, is instructive. For this approach a special notation is introduced. Let

J be the probability that Jones survives, S the probability that Smith survives, J' the probability that Jones dies, and S' the probability that Smith dies. Then

$$J = 0.8, \qquad J' = 0.2;$$
$$S = 0.9, \qquad S' = 0.1.$$

There are only four possible events. They are mutually exclusive, and one is sure to happen. The probabilities of these four events are:

$$\begin{aligned} JS &= 0.8 \times 0.9 = 0.72 \\ JS' &= 0.8 \times 0.1 = 0.08 \\ J'S &= 0.2 \times 0.9 = 0.18 \\ J'S' &= 0.2 \times 0.1 = \underline{0.02} \\ & \qquad\qquad\quad\; 1.00 \end{aligned}$$

The probability that at least one dies is $JS' + J'S + J'S' = 0.08 + 0.18 + 0.02 = 0.28$.

Example 3

Six persons toss one coin for a prize which is to be won by the one who first throws a head. If they throw in succession, what is the probability that the fourth person will win?

Solution: The fourth person could conceivably win on either his first throw, his second throw, his third throw, etc. In order for him to win on his first throw the three persons ahead of him must all throw tails after which the fourth person throws a head. If nine consecutive tails are thrown and the tenth throw is a head, the fourth person wins on his second try. And so on. Since a head is just as likely as a tail on any single throw, we have from theorem III that the probability of the fourth person's winning on his first throw is $(\frac{1}{2})^4$. Similarly, the probability of his winning on his second throw is $(\frac{1}{2})^{10}$, on his third throw $(\frac{1}{2})^{16}$, and so on. From theorem I the required probability is

$$(\tfrac{1}{2})^4 + (\tfrac{1}{2})^{10} + (\tfrac{1}{2})^{16} + (\tfrac{1}{2})^{22} + \cdots.$$

This is an infinite geometric progression whose first term is $(\frac{1}{2})^4$ and whose ratio is $(\frac{1}{2})^6$. Recalling that $S = a/(1 - r)$ where S is the sum of an infinite geometric progression whose first term is a and whose ratio is r, we have the required probability equal to

$$\frac{(\frac{1}{2})^4}{1 - (\frac{1}{2})^6} = \frac{2^2}{2^6 - 1} = \frac{4}{63}.$$

PROBLEM SET 1

1. What is the probability of getting an 8 in a throw of 2 ordinary dice?

2. One die is thrown twice. What is the probability of a 4 on the first throw and a 3 on the second? 4 on both throws?

3. A bag contains 5 white, 6 red, and 4 black balls. If a ball is drawn at random, what is the probability that it is a white ball? a black or white ball?

4. If 3 balls are drawn from the bag in problem 3, what is the chance that they are all white?

5. Three cards are drawn from an ordinary deck. What is the chance that all are aces? that an ace, a king, and a queen are drawn in any order?

6. If a bag contains 5 black and 3 white balls and a second bag contains 3 black and 7 white balls, what is the probability of drawing 1 black and 1 white ball in 2 drawings, 1 from each bag?

7. Five coins are tossed. What is the probability of obtaining exactly 3 heads? at least 3 heads? not more than 3 heads?

8. Brown and Schultz play chess. On the basis of past performances, Brown is a 3 to 2 favorite to win any given game that is not drawn. What is the probability that Schultz will win 2 or more out of a series of 3 if a drawn game is replayed?

9. Three dice are thrown. What is the probability that the sum of the faces is 7?

10. Four dice are thrown. What is the probability that the sum of the faces is 9?

11. The probability that an individual aged 20 will live 20 years is 0.9, and the probability that an individual aged 40 will live 10 years is 0.8. What is the probability that an individual aged 20 will (*a*) live 30 years? (*b*) die before age 50? (*c*) die between ages 40 and 50?

12. The probabilities that A, B, C, and D will die during a certain period are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$, respectively. What is the probability that (*a*) all four die during the period? (*b*) all four survive? (*c*) at least one dies? (*d*) at least one survives? (*e*) exactly two die?

13. The probability that a man aged 20 and another man aged 40 will both survive a period of 20 years is 0.6. Out of 50,000 men alive at age 20, [illegible] will die before attaining age 25. Calculate the probability that a man now aged 25 will die within the next 35 years.

14. A and B take turns throwing a coin, A throwing first. The first one to throw a head wins. Find the probability of winning for each.

15. According to records in the registrar's office, 5% of students fail a certain course. What is the probability that in a group of 6 students, picked at random, exactly 2 fail?

16. In each of a set of games it is 2 to 1 in favor of the winner of the previous game. What is the chance that the player who wins the first game shall win at least 3 of the next 4?

17. The probability that a man aged x will die in a year is $\frac{1}{10}$. Find the probability that out of 4 men, A, B, C, D, each aged x, A will die in the year and be the first to die.

18. A mortality study indicates that out of 100 males born at the same time 1 dies annually until there are no survivors. If 3 men were known to be alive 5 years ago when their ages were 20, 30, and 60, find the chance that all are now alive.

19. The probability that exactly 1 life out of 3 lives aged 20, 35, and 50 will survive 15 years is 0.092; the probability that all will die within 15 years is 0.006. If the probability that a life aged 20 will die before age 35 is 0.1, find the probability that he will live to age 65.

20. Nine balls are marked 1, 2, 3, · · · , 9 and placed in a bag. Three balls are drawn at random from the bag. Find the probability that the sum of the markings is divisible by 3.

6 · THE MORTALITY TABLE

It was mentioned in section 1 that life insurance calculations depend on three major factors and that one of these factors is the probability that a given person will die in a given period of time. A convenient way of measuring probabilities of living or dying is essential if these calculations are to be performed efficiently. The mortality table, being in effect a table of probabilities, fills this need. In this section we shall confine ourselves to finding out what a mortality table is and how it can be used to calculate probabilities of living and dying. In section 7 we shall discuss briefly how a mortality table is constructed.

Let l_0 represent the number of persons who, according to the mortality table, are born in any year of time. Some of these l_0 persons will die before their first birthday. Others will still be alive on their first birthday. Let l_1 represent the number of the l_0 original persons who reach age 1. Now some of these l_1 persons will die before age 2. Others will live to attain age 2. Let l_2 represent the number of survivors who attain age 2. It is important to observe that we can think of l_2 as arising in either of two ways as follows: (*a*) There were l_0 persons who started the race of life at age 0. Of these l_0 persons, l_2 lived to see their second birthday. (*b*) There were l_1 persons alive at age 1. Of these l_1 persons, l_2 lived to see their second birthday.

By continuing this line of thought we arrive at the general definition that l_x represents the number of persons who, according to the mortality table, attain precise age x in any year of time. The actual numbers represented by l_0, l_1, l_2, etc., are different for different mortality tables. The reader should understand that, although modern mortality tables are based on a careful study of available data, in the last analysis any mortality table is necessarily somewhat arbitrary. The sources and characteristics of some of the more important mortality tables are discussed in the next section. The column from the Commissioners 1941 Standard Ordinary Mortality Table (popularly called the CSO table) is shown in Table 2 in Appendix 3. Note that $l_{100} = 0$. This means that, according to the data and theory underlying the CSO table, the chance that anyone will live to be 100 is so small that

it may be ignored for practical purposes. The greatest age for which $l_x > 0$ is denoted by ω. $\omega + 1$ is called the limiting age of the table. For the CSO table $\omega = 99$, and the limiting age is 100.

From this l_x column we can calculate the probabilities that a person aged x will live to attain some greater age or that he will die between any two birthdays. The following examples show the method of calculation:

Example 1

What is the probability that a person aged 30 will attain age 50?

Solution: Of the l_{30} persons alive at age 30, l_{50} will still be alive at age 50. Therefore, the required probability is l_{50}/l_{30}. Assuming that mortality will follow the CSO table, this probability becomes $\frac{810{,}900}{924{,}609} = 0.87702$. By referring to Table 2 the reader can easily verify that $l_{50} = 810{,}900$ and $l_{30} = 924{,}609$.

Example 2

What is the probability that a person aged 20 will die within 1 year?

Solution: Of the l_{20} persons alive at age 20, l_{21} will still be alive at age 21. Therefore $l_{20} - l_{21}$ persons will die within 1 year, so that the required probability is

$$\frac{l_{20} - l_{21}}{l_{20}} = \frac{951{,}483 - 949{,}171}{951{,}483} = 0.00243.$$

Example 3

What is the probability that a person aged 25 will die between ages 60 and 70?

Solution: Of the l_{25} persons alive at age 25, l_{60} will be alive at age 60 and l_{70} at age 70. Hence, of the l_{25} persons, $l_{60} - l_{70}$ will die between ages 60 and 70 so that the required probability is

$$\frac{l_{60} - l_{70}}{l_{25}} = \frac{677{,}771 - 454{,}548}{939{,}197} = 0.23767.$$

Although the l_x column is all we need to compute the probabilities used in this book, it is convenient to have another column called the d_x column. The term d_x represents the number, out of the l_x persons attaining precise age x, who die before reaching age $x + 1$. Thus,

$$d_x = l_x - l_{x+1}. \tag{1}$$

We could have used the d_x column to advantage in solving example 2 above. Thus, of the l_{20} persons alive at age 20, d_{20} will die within 1 year so that the required probability is

$$\frac{d_{20}}{l_{20}} = \frac{2312}{951{,}483} = 0.00243.$$

It is important to note that no number in the l_x or d_x columns means anything by itself. It is the ratios that have meaning. We could multiply every number in the two columns by, say, 15 without changing the mortality table in any fundamental respect.

PROBLEM SET 2

Solve the following problems by assuming that mortality follows the CSO table. Compute answers to 3 decimals.

1. What is the probability that a 1-year-old child will live to be 70?
2. What is the probability that a 1-year-old child will die between ages 50 and 60?
3. A couple has 2 children, age 1 and 11. Find the probability that exactly 1 of them dies before age 50.

7 • SOURCES AND CHARACTERISTICS OF MORTALITY TABLES

In constructing a mortality table the first step is to make a study whose object is a schedule showing the number of persons exposed to the risk of death at each age and the number of such persons who died at that age. The ratio of the number of deaths to the number exposed is the rate of mortality; that is, the probability that a person who has attained a certain age will die within a year. The next step is to choose a convenient arbitrary number, called the radix of the table, to represent the number of persons living at the lowest age in the investigation. By starting with this radix we can use the rates of mortality to build the l_x and d_x columns of the mortality table.

Example 1

It is desired to construct a mortality table for the period represented by ages 18 to 23 only. An investigation produces the following data:

Age	*Number Exposed*	*Number of Deaths*
18	5,000	10
19	10,000	22
20	15,000	36
21	10,000	27
22	20,000	60

Dividing the number of deaths at each age by the corresponding number exposed we have

Age	*Probability of Dying within 1 Year*
18	0.0020
19	0.0022
20	0.0024
21	0.0027
22	0.0030

Now let $l_{18} = 100{,}000$ (the radix). Then d_{18} is the number of persons alive at age 18 times the probability of death between ages 18 and 19. That is, $d_{18} = 100{,}000 \times 0.002 = 200$.

$$l_{19} = l_{18} - d_{18} = 100{,}000 - 200 = 99{,}800.$$
$$d_{19} = 99{,}800 \times 0.0022 = 220 \quad \text{(to the nearest integer)}.$$
$$l_{20} = 99{,}800 - 220 = 99{,}580.$$

Continuing, we have the following mortality table:

x	l_x	d_x
18	100,000	200
19	99,800	220
20	99,580	239
21	99,341	268
22	99,073	297
23	98,776	

Actually, the problems involved in constructing a mortality table are highly technical and beyond the scope of this textbook. The above example has been deliberately simplified in order to give the reader a general idea of how a mortality table is constructed.

Many important mortality tables have been based on the experience of the general population, the basic data coming from an official census. Since life insurance companies do not ordinarily issue life insurance policies to persons not in reasonably good health, the rates of mortality in a table constructed from census records would be higher than the rates of mortality in a table constructed from insurance company records, provided the two tables were based on records covering roughly the same period of time. Accordingly, life insurance calculations are based on the experience of life insurance policyholders.

Four of the better-known life insurance mortality tables in this country are the American Experience Table, the American Men Table, Table Z, and the CSO Table.

THE AMERICAN EXPERIENCE TABLE. The American Experience Table was constructed about 1860 and first published under its present name in 1868. Sheppard Homans, an actuary affiliated with the Mutual Life Insurance Company of New York and the author of the table, never gave full particulars as to how the table was constructed. He once stated publicly, however, that, although the experience of the Mutual Life Insurance Company of New York was the main basis for the table, the table was never intended to be an accurate interpretation of the experience of the Mutual Life. It seems reasonable to conclude that Mr. Homans' judgment played a large part in the construction of the table. The table starts at age 10 with a radix of 100,000 and ends with three deaths between ages 95 and 96 so that $\omega = 95$ and 96 is the limiting age of the table.

Until very recently the American Experience Table was very widely used for premium and reserve calculations. It is still the basis upon which a large proportion of outstanding insurance in force was issued, and many laws and regulations refer to it. It is, therefore, of considerably more than academic interest.

THE AMERICAN MEN TABLE. The American Men Table was published in 1918 by the Actuarial Society of America with the cooperation of the American Institute of Actuaries and the National Convention of Insurance Commissioners. The basic data was the experience of life insurance companies for the period of 1900 to 1915, inclusive. Rates of mortality were produced not only for each age but also for various durations since the date of issue of the insurance policy. It may be argued, without benefit of statistics, that a group of persons aged 40 who have just qualified for life insurance are "select" lives and that the probabilities of dying in the next year are less for such a group than for a group of persons aged 40 who qualified for insurance some years ago. Many of the latter group would not now be eligible for insurance. This general reasoning argument is upheld by life insurance statistics, which allow us to conclude that the rate of mortality at a given age increases with duration because the effects of the initial selection gradually wear off. The studies made in connection with the American Men Table indicated that for practical purposes the effects of selection could be assumed to disappear after 5 years. This means, for example, that the rate of mortality for age 40, duration 10, is just about the same as for age 40, duration 15—both being greater than for age 40, duration 3.

The American Men Table was accordingly published with a 5 year select period. The following extract from this table shows the rates of mortality for selected ages.

AMERICAN MEN MORTALITY TABLE
RATE OF MORTALITY PER 1000

Age at Issue [x]	Year of Insurance						Attained Age
	1	2	3	4	5	6 and over	
[15]	2.47	3.24	3.41	3.55	3.72	3.92	20
[16]	2.52	3.31	3.48	3.63	3.82	4.02	21
[17]	2.56	3.37	3.55	3.73	3.92	4.12	22
[18]	2.61	3.44	3.64	3.81	4.00	4.18	23
[19]	2.66	3.52	3.72	3.89	4.07	4.25	24
[20]	2.73	3.59	3.80	3.96	4.13	4.31	25
[21]	2.78	3.66	3.86	4.01	4.18	4.35	26
[22]	2.83	3.72	3.91	4.06	4.21	4.39	27
[23]	2.86	3.76	3.96	4.08	4.24	4.41	28
[24]	2.91	3.80	3.99	4.11	4.26	4.43	29
[25]	2.93	3.84	4.02	4.12	4.27	4.46	30
[26]	2.95	3.86	4.04	4.13	4.28	4.48	31
[27]	2.98	3.88	4.06	4.14	4.29	4.51	32
[28]	2.98	3.91	4.06	4.14	4.32	4.59	33
[29]	2.99	3.92	4.08	4.17	4.37	4.68	34

In the above table the figure 3.80 for age at issue 20, year of insurance 3, means that 0.00380 is the probability that a person now aged 22 who was accepted for life insurance at age 20 will die before attaining age 23. Note that the year of insurance is one more than the duration at the start of the year. In this example the first year of insurance begins at age 20 (duration 0); the second year of insurance begins at age 21 (duration 1); the third year of insurance begins at age 22 (duration 2); etc. The figure 4.31 for attained age 25 means that 0.00431 is the probability that a person now aged 25 who was accepted for insurance at least 5 years ago will die before attaining age 26. Note that the age at issue governs only the first five columns and that the sixth column is governed by attained age.

The sixth column represents the experience after the effect of selection is considered to have disappeared, and these rates of mortality are called "ultimate" rates. A table based on ultimate rates is called an ultimate table. A table showing the rate of mortality by both age and duration is called a select table. The American Experience Table, Table Z, and the CSO Table are ultimate tables.

The American Men (Ultimate) Table is the basis for most group life insurance issued in this country today. It was also used by several

important companies as the basis of premiums and reserves before the adoption of the CSO Table.

EXAMPLE 2

Use the American Men (Select) Table to calculate the following probabilities. Do not perform the actual multiplication.

(*a*) What is the probability that a man now aged 20 who was accepted for life insurance a year ago will die between ages 21 and 22? (*b*) between ages 26 and 27?

Solution: (*a*) The required probability is the probability that he will live to be 21 times the probability that, having lived to 21, he will die in the next year. That is,

$$\begin{aligned}\text{Required probability} &= (1 - 0.00352)(0.00372)\\ &= (0.99648)(0.00372).\end{aligned}$$

(*b*) Following the same line of reasoning as in (*a*), we have for the required probability

$$(0.99648)(0.99628)(0.99611)(0.99593)(0.99569)(0.00435).$$

TABLE Z. Table Z was based on life insurance experience for the period 1920 to 1934, inclusive, and was prepared in 1939. The original purpose of the table was to provide a measure of representative modern mortality experience for purposes of comparison. It has been used, however, by a number of companies as the basis for nonparticipating insurance premiums. When so used, a select period has usually been grafted on. The subject of nonparticipating premiums is discussed in Chapter 9. In a word, the difference between nonparticipating premiums and participating premiums is that the latter contemplate a refund (dividend) to the policyholder in case the premium proves to be more than sufficient to fulfill the company's obligations under the insurance contract whereas nonparticipating premiums contemplate no such refund. Nonparticipating premiums are, of course, generally lower for comparable insurance benefits.

CSO TABLE. The CSO Table was based on life insurance experience for the period 1930 to 1940, inclusive. The observed rates of mortality were arbitrarily increased in order to provide the reasonable safety margin necessary to the sound operation of the life insurance business. The table is shown in full in Table 2 in Appendix 3. Examples and problems in this book assume CSO mortality unless otherwise specified.

The CSO Table is the basis for most life insurance currently issued in this country. Many contracts now in force were issued before the adoption of the CSO Table and are, therefore, based on older tables.

As mentioned above, the most important of these older tables is the American Experience.

ANNUITIES. The statistical evidence is conclusive that (*a*) the mortality rates for annuitants are lower than for holders of life insurance policies and (*b*) the mortality rates for female annuitants are less than for male annuitants at corresponding ages. It is, therefore, unsound to base annuity calculations on an insurance mortality table. In the past, various annuity mortality tables have been used. The most important are the McClintock Table (1886), the American Annuitants Table (1920), the Combined Annuity Table (1928), and the 1937 Standard Annuity Table. Because annuitant mortality has shown continued improvement over the years, all these tables, except the 1937 Standard, are out of date as bases for currently issued annuities. The 1937 Standard, instead of having separate mortality tables for males and females, assumes that the female table is the male table set back 5 years. This means that the mortality rate for a female is assumed to be the same as for a male 5 years younger. Because of improving mortality, annuity calculations are sometimes made on the basis of the 1937 Standard Annuity Table set back 1 or 2 years for both males and females. A setback of 1 year for females is really a 6-year setback on the table because the 1-year setback is in addition to the customary 5-year setback for females.

It is very important that the reader appreciate the fundamental difference between a conservative insurance mortality table and a conservative annuity mortality table. An insurance table is conservative if the rates of mortality according to the table are *greater* than the rates actually experienced. An annuity table is conservative if tbe rates of mortality according to the table are *less* than the rates actually experienced. Since mortality rates have been decreasing for both annuitants and holders of life insurance policies and since there is every reason to believe that these rates will continue to decrease, it follows that passage of time causes an insurance table to become more conservative and an annuity table to become less conservative.

Thus, the 1937 Standard Annuity Table, although in general use and specifically approved by the laws of most states, is already considered out of date by many persons. At a meeting of the Society of Actuaries in November 1949, W. A. Jenkins and E. A. Lew presented a paper entitled "A New Mortality Basis for Annuities."* The

* At the present writing this paper has not been published. It is expected, however, that it will be included in Volume I of the *Transactions* of the Society of Actuaries.

basis described in this paper embodies a new idea which is beyond the scope of this book. For our present purpose it is sufficient to state that this new basis will probably eventually supersede the 1937 Standard Annuity Table.

The following table shows the rates of mortality at various ages for the five tables discussed:

RATE OF MORTALITY PER 1000

Age	American Experience	American Men Ultimate	CSO	Table Z	1937 Standard Annuity	
					Male	Female
20	7.81	3.92	2.43	2.23	1.33	1.26
25	8.07	4.31	2.88	2.37	1.56	1.33
30	8.43	4.46	3.56	2.52	2.07	1.56
35	8.95	4.78	4.59	3.15	2.98	2.07
40	9.79	5.84	6.18	4.51	4.36	2.98
45	11.16	7.94	8.61	6.93	6.36	4.36
50	13.78	11.58	12.32	10.14	9.29	6.36
55	18.57	17.47	17.98	15.30	13.55	9.29
60	26.69	26.68	26.59	23.37	19.75	13.55
65	40.13	40.66	39.64	35.62	28.75	19.75
70	61.99	61.47	59.30	53.94	41.76	28.75
75	94.37	91.94	88.64	83.20	60.46	41.76
80	144.47	135.74	131.85	127.13	87.16	60.46

Earlier in this section, under the heading "CSO Table," it was mentioned that examples and problems in this book assume CSO mortality unless otherwise specified. Under the heading "Annuities," the statement was made that it is unsound to base annuity calculations on an insurance mortality table. These two statements may appear inconsistent to the reader. The apparent inconsistency can be resolved only by an appreciation of the major purpose of this book, which is to consider methods of calculating elementary life insurance functions using an ultimate table. Since the mathematical principles are the same for all such tables, working examples based on a second table will add nothing to the reader's technique.

There is, however, some danger in basing all examples on one table. The reader may lose sight of the fact that different tables are used for different purposes. In particular, it is important for the reader to appreciate the difference between an insurance table and an annuity table. Accordingly, several examples and problems in this book are based on the 1937 Standard Annuity Table. When this table is to

be used, the fact will be clearly stated in the problem. The 1937 Standard Annuity Table is shown in Table 14 of Appendix 3.

PROBLEM SET 3

Use the American Men (Select) Table to solve problems 1 through 4. Do not perform tbe actual multiplication.

1. What is tbe probability tbat a man now aged 19 who was accepted for life insurance 2 years ago will live to be 21?

2. What is the probability that a man now aged 25 who was accepted for life insurance 3 years ago will die between ages 30 and 31?

3. What is the probability that a man now aged 26 who was accepted for life insurance at age 15 will die in tbe next year?

4. Consider two men, botb aged 20. One has just been accepted for life insurance, the otber was accepted 10 years ago. Find tbe probability that at least one will die before age 22.

5. Complete tbe following table:

x	*Rate of Mortality*	l_x	d_x
95	1/3	1500	
96	2/5		
97	1/2		
98	2/3		
99	1		
100			

8 • NOTATION

The symbol (x) is used to represent a person aged x, sometimes also called a life aged x.

${}_np_x$ denotes the probability that (x) will live n years. ${}_np_x = l_{x+n}/l_x$. If $n = 1$, the left-hand subscript is omitted. Thus, p_x denotes the probability that (x) will live 1 year. $p_x = l_{x+1}/l_x$.

${}_nq_x$ denotes the probability that (x) will die within n years.

$$ {}_nq_x = \frac{l_x - l_{x+n}}{l_x}. $$

If $n = 1$, the left-hand subscript is omitted. Thus, q_x denotes the probability that (x) will die within 1 year.

$$ q_x = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_x}. $$

${}_{m|n}q_x$ denotes the probability that (x) will live m years but die in the next n years; that is, that (x) will die between ages $x + m$ and $x + m + n$.

$$ {}_{m|n}q_x = \frac{l_{x+m} - l_{x+m+n}}{l_x}. $$

If $n = 1$, it is omitted from the notation so that ${}_{m|}q_x$ denotes the probability that (x) dies between ages $x + m$ and $x + m + 1$.

$$ {}_{m|}q_x = \frac{l_{x+m} - l_{x+m+1}}{l_x} = \frac{d_{x+m}}{l_x}. $$

The following general observations should be made:

The right-hand subscript represents the present age of the person in question.

The left-hand subscript represents the period of years during which the event (living or dying) is to take place.

The letter or number before the bar represents the period of deferment.

The letter p is used to denote the probability of an individual's living a given period.

The letter q is used to denote the probability of an individual's dying during a given period.

NUMERICAL ILLUSTRATIONS

${}_{10}p_{20}$ represents the probability that a life aged 20 will live 10 years—i.e., will live to be age 30.

p_{20} represents the probability that (20) will live to be 21.

${}_5q_{30}$ represents the probability that (30) will die before reaching age 35.

q_{30} represents the probability that (30) will die before reaching age 31.

${}_{10|5}q_{20}$ represents the probability that (20) will die between ages 30 and 35; that is, that (20) will live 10 years but die during the next 5.

${}_{10|}q_{20}$ represents the probability that (20) will die between ages 30 and 31.

9 • EXPECTATION OF LIFE

The average number of complete years to be lived in the future by persons now aged x is called the curtate expectation of life of (x). ("Shortened" is the dictionary definition of curtate. The word is rarely used except in connection with expectation of life.) The words "complete years" imply that, in computing the average, fractions of a year lived will be disregarded. This is the same as assuming that all deaths occur in the instant after a birthday.

Consider l_x persons who are alive at age x. l_{x+1} of these will be alive at age $x + 1$, so that a total of l_{x+1} years will be lived during the first year following the reference age, x. Similarly, a total of l_{x+2} years will be lived during the second year, l_{x+3} during the third

year, and so forth. The total number of years lived by these l_x persons will be $l_{x+1} + l_{x+2} + l_{x+3} + \cdots + l_\omega$, so that the average number of full years, or the curtate expectation of life, is

$$e_x = \frac{l_{x+1} + l_{x+2} + l_{x+3} + \cdots + l_\omega}{l_x}. \tag{2}$$

This question can be looked at in another way, by using the concept of mathematical expectation. If we think of (x) as receiving a reward of 1 for every complete year that he lives, his mathematical expectation for the whole of life* would be the probability that he lives 1 year times his reward of 1 plus the probability that he lives 2 years times his reward of 1, and so forth. In symbols,

$$e_x = p_x + {}_2p_x + {}_3p_x + \cdots + {}_{\omega-x}p_x.$$

It is shown algebraically in example 1, section 10, that the two expressions for e_x are exactly equal.

The average number of years, including fractions, to be lived in the future by persons now aged x is called the complete expectation of life of (x) and is denoted by $\mathring{e}_x$. If we assume that deaths are evenly distributed throughout the year, we can say that on the average a person will be halfway from one birthday to the next at death. Therefore

$$\mathring{e}_x = e_x + \tfrac{1}{2} \quad \text{approximately.} \tag{3}$$

The word "approximately" is necessary because of our assumption that deaths are evenly distributed over any given year of age.

The concept of expectation of life is useful in two major respects. First, it is useful in comparing in very general terms the level of mortality under various mortality tables. Second, it can be used by experienced actuaries in making a rough analysis of certain types of problems. Contrary to rather popular opinion, this concept is not used in precise actuarial calculations relating to premiums.

PROBLEM SET 4

1. Find the numerical values of each of the following probabilities from the CSO Table, but do not perform the actual division. Give the proper symbol in each case.

(*a*) The probability that (25) will live to age 60.

(*b*) The probability that (20) will live at least 60 more years.

(*c*) The probability that (27) will die within a year after his forty-first birthday.

(*d*) The probability that (65) will live at least 10 but not more than 15 years more.

(*e*) The probability that both (18) and (19) will live 50 more years.

---

* His mathematical expectation, considering the first 10 years only, would be $p_x + {}_2p_x + {}_3p_x + \cdots + {}_{10}p_x$.

2. State in words the probabilities represented by the following symbols:

$$p_{40},\ {}_{10}p_{23},\ q_{40},\ {}_{20}q_{40},\ {}_{10|20}q_{30},\ {}_{10|}q_{30}.$$

3. Between what two consecutive ages is a man now 21 most likely to die? What is the probability that he will die in that year?

4. What is the curtate expectation of (21)? What is the probability that (21) will live at least that long?

5. Complete the following table, calculating q_x and p_x to 4 decimals, and e_x and $\mathring{e}_x$ to 2 decimals.

x	l_x	d_x	q_x	p_x	e_x	$\mathring{e}_x$
95	1000					
96	700					
97	400					
98	100					
99	10					
100	0					

6. If $l_x = 200(100 - x)$, find ${}_{2|}q_x$.

7. Compute to 3 decimals the probability that a man aged 65 will live at least 15 more years.

8. Compute to 3 decimals the probability that a woman aged 65 will live at least 15 more years.

9. Work problem 7 on the assumption of 1937 Standard Annuity mortality.

10. Work problem 8 on the assumption of 1937 Standard Annuity mortality.

10 • IDENTITIES

The reader will recall from algebra that an identity is an equation that is satisfied by all possible values of the variable or variables which the equation contains. The equation

$$ {}_np_x = \frac{l_{x+n}}{l_x}$$

is an example of an identity based on the mortality table. It is an identity because it is satisfied by all values of x and n. For example, if $n = 10$ and $x = 25$, the relation becomes

$$ {}_{10}p_{25} = \frac{l_{35}}{l_{25}}.$$

Or, if we write $m + n$ instead of n and $x - n$ instead of x, the relation becomes

$$ {}_{m+n}p_{x-n} = \frac{l_{x-n+m+n}}{l_{x-n}} = \frac{l_{x+m}}{l_{x-n}}.$$

Scattered throughout this book are a number of exercises in which the reader will be asked to prove an identity. Such exercises are valuable for the following three major reasons:

(*a*) Proving an identity is a very useful exercise in mathematical reasoning and algebraic manipulation.

(*b*) Proving identities tends to clarify the meaning of the various symbols and the relationships among them.

(*c*) A person who is adept at transforming one expression into another equivalent expression can frequently save hours of calculating time in performing calculations necessary in practical actuarial work.

The problems involved in proving identities in life contingencies are very similar to the problems involved in trigonometric identities. It is impossible to give a general method of proof. Ordinarily, in proving an identity, we must transform one side of the identity into the other. It is usually better to start with the more complicated side and try to reduce ~~it to the form~~ of the simpler side. In problems that do not involve interest it frequently works to start by changing the more complicated side into the l's and d's of the mortality table. If interest is involved, the identity can frequently be easily proved by changing the more complicated side into commutation symbols and proceeding from there. (Commutation symbols are introduced in Chapter 3.)

Persons with a feeling for mathematical language tend to be a little fussy about the form of a proof. In this connection it is well to remember the following points:

(*a*) Mathematical notation is nothing but a shorthand form of English—or whatever the individual's native tongue may be.

(*b*) A good proof should read like a sentence or paragraph in English.

(*c*) In attempting to prove an identity, an equals sign should never be used without qualification unless the user really means "equals." Putting an unqualified "=" between the two sides of the identity, before the identity has actually been established, is generally conceded to be poor form.

(*d*) Enough steps should be shown so that the development can be followed by anyone who is reasonably familiar with the subject.

EXAMPLE 1

Show that

$$e_x = p_x + {}_2p_x + {}_3p_x + \cdots + {}_{\omega-x}p_x.$$

Solution:

$$p_x + {}_2p_x + {}_3p_x + \cdots + {}_{\omega-x}p_x = \frac{l_{x+1}}{l_x} + \frac{l_{x+2}}{l_x} + \frac{l_{x+3}}{l_x} + \cdots + \frac{l_\omega}{l_x}$$

$$= \frac{l_{x+1} + l_{x+2} + l_{x+3} + \cdots + l_\omega}{l_x}$$

$$= e_x \quad \text{by formula 2.}$$

Example 2

Show that

$$l_x - l_{x+n} = d_x + d_{x+1} + d_{x+2} + \cdots + d_{x+n-1}$$

Solution:

$$\begin{aligned} &d_x + d_{x+1} + d_{x+2} + \cdots + d_{x+n-1} \\ &\quad = (l_x - l_{x+1}) + (l_{x+1} - l_{x+2}) + (l_{x+2} - l_{x+3}) + \cdots \\ &\qquad + (l_{x+n-2} - l_{x+n-1}) + (l_{x+n-1} - l_{x+n}) \\ &\quad = l_x - l_{x+1} + l_{x+1} - l_{x+2} + l_{x+2} - l_{x+3} + \cdots \\ &\qquad + l_{x+n-2} - l_{x+n-1} + l_{x+n-1} - l_{x+n} \\ &\quad = l_x - l_{x+n}, \end{aligned}$$

since all other terms cancel by pairs.

Example 3

Show that

$${}_{m+n}p_x = {}_mp_x - {}_{m|}{}_nq_x.$$

Solution:

$${}_mp_x - {}_{m|}{}_nq_x = \frac{l_{x+m}}{l_x} - \frac{l_{x+m} - l_{x+m+n}}{l_x}$$

$$= \frac{l_{x+m} - l_{x+m} + l_{x+m+n}}{l_x}$$

$$= \frac{l_{x+m+n}}{l_x}$$

$$= {}_{m+n}p_x.$$

PROBLEM SET 5

Prove the following identities:

1. $e_x = p_x(1 + e_{x+1})$.
2. ${}_{m+n}p_x = {}_mp_x \cdot {}_np_{x+m}$.
3. ${}_3p_x = p_x \cdot p_{x+1} \cdot p_{x+2}$.
4. ${}_np_x = p_x \cdot p_{x+1} \cdot p_{x+2} \cdots p_{x+n-1}$.

5. $_{m|n}q_x = {}_mp_x \cdot {}_nq_{x+m}$.

6. $$\frac{e_x \cdot e_{x+1} \cdot e_{x+2} \cdots e_{x+n-1}}{(1+e_{x+1})(1+e_{x+2})(1+e_{x+3}) \cdots (1+e_{x+n})} = {}_np_x.$$

7. $q_x + p_x \cdot q_{x+1} + {}_2p_x \cdot q_{x+2} + \cdots = 1$.

8. $${}_np_x = \frac{{}_np_x - {}_{n+1}p_x}{q_{x+n}}.$$

PROBLEM SET 6

1. Three dice are thrown. Find the probability that the sum of the faces is at least 7.

2. Three cards are drawn from an ordinary deck. Find the probability that at least one of the cards is an ace, king, or queen.

3. Johnson, Schwartz, and Riley draw numbere from 1 to 10 for a prize of \$10.80, to be won by the man making the highest draw. Johnson has drawn a 6. Find the expectation of each man.

4. In a certain city it is found that out of every 2000 houses 9 are destroyed annually by fire. In the absence of any other information, find what premium a fire insurance company should charge for insuring a house for \$6000 against a total loss, if no allowance is made for the expense of doing business.

5. Five cards are drawn from an ordinary deck. Find the probability that they are all spades.

6. The probability that an individual aged 18 will live 10 years is 0.95, and the probability that the individual will live 30 years is 0.75. Find the probability that an individual aged 28 will die before reaching age 48.

7. Using the CSO Table and computing final answers to 4 decimals, solve the following problems:

(*a*) What is the probability that a 1-year-old child will die between ages 20 and 30?

(*b*) What is the probability that a 1-year-old child will die between ages 20 and 21?

(*c*) A man and woman are married when each is 25 years old. On their silver wedding they have two children, ages 20 and 24. Find the probability that all four will be alive on the parents' golden wedding.

8. Using the American Men (Select) Table, find the probability that a man now aged 21 who was accepted for life insurance 3 years ago will die between ages 22 and 25.

9. State in words the probabilities represented by the following symbols:

$$p_{33},\ {}_{10}p_{33},\ q_{33},\ {}_{20}q_{33},\ {}_{10|20}q_{33},\ {}_{10|}q_{30}.$$

10. If $l_x = k(185 - 2x)$, find p_{85}.

11. Work problem 7c on the assumption of 1937 Standard Annuity mortality. Assume that both children are male.

CHAPTER TWO

Interest and Annuities Certain

11 · INTEREST

Interest is the payment the borrower makes for the use of the lender's money. A sum of money placed at interest is called the *principal*. The *rate* of interest is the amount earned in 1 year by a principal of $1 (technically, 1, rather than $1, since there is no unit in the rate of interest). The rate of interest is customarily expressed in percentage terms.

Almost everyone is familiar with the concept of simple interest. If $100 is borrowed for a year at 6% interest, the amount due at the end of the year is the principal ($100) plus the interest (6% of $100), or $106 in all. In general, a principal P loaned for a year at a rate i will earn interest of Pi and will amount at the end of the year to

$$P + Pi = P(1 + i).$$

12 · COMPOUND INTEREST

The whole structure of life insurance is based on the assumption that the insurance company's funds are constantly invested and that the investments are earning interest. The word "constantly" implies that, when the insurance company receives an interest payment from a borrower, the interest payment is immediately invested and begins to earn more interest. In other words, the theory of life insurance assumes compound interest.

In section 11 it was pointed out that a principal P loaned for a year at a rate i will amount to $P(1 + i)$ at the end of the year. If this new principal of $P(1 + i)$ is immediately invested at the rate i, the interest due at the end of another year will be $P(1 + i)i$, and the total amount will be $P(1 + i) + P(1 + i)i = P(1 + i)^2$. In general, if P is invested at a rate i and if each interest payment is immediately invested at the same rate i, the total accumulated amount at the end of n years will be $P(1 + i)^n$.

It is frequently necessary to answer the question, "What is the present value of an amount A due n years from now?" This is the same thing as asking, "What sum invested now at compound interest will amount to A in n years?" The solution is very simple. Let P be the principal that must be invested now. After n years at compound interest this will amount to $P(1 + i)^n$. But according to the conditions of the problem this amount must be equal to A. That is, we have the equation

$$P(1 + i)^n = A,$$

from which

$$P = A(1 + i)^{-n}.$$

The symbol v is defined as $(1 + i)^{-1}$; hence P can be written as Av^n. If A happens to be 1, $P = v^n$. That is to say that the present value of 1 due in n years is v^n.

Tables of $(1 + i)^n$ and v^n at $2\frac{1}{2}\%$ interest are given in Table 1 in Appendix 3.

Discount. In many types of loans, particularly small loans made by banks, the interest charge is deducted in advance. This charge is usually called either *interest in advance* or *discount.* For example, under a \$100 loan with 6% interest in advance the borrower actually receives \$94 and agrees to pay \$100 at the end of 1 year. In general, the rate of discount d is defined as the deduction divided by the amount due at the end of the year. Thus, on a loan of \$1 at discount rate d the actual amount of the loan, or principal, is $1 - d$, the amount of interest is d, and the rate of interest is given by

$$i = \frac{d}{1 - d},$$

since the rate of interest is defined as the amount of interest divided by the principal.

Solving for d, we have

$$i(1 - d) = d,$$

$$i - id = d,$$

$$i = id + d,$$

$$d = \frac{i}{1 + i},$$

or

$$d = iv, \text{ since } v = \frac{1}{1+i}.$$

The concept of discount is very useful in life insurance because of the simplification permitted in various problems.

13 · ANNUITY CERTAIN

An annuity certain is a series of periodic payments payable over a fixed period of time, it being assumed that each payment is certain to be made when it is due. The payments may or may not be equal size. In order to simplify the discussion at this point, this chapter will be confined to annuities certain in which payments of equal size are made annually. The word "certain" in the term "annuity certain" is used to distinguish an annuity for which it is assumed that the payments are certain to be made when due from a life annuity, where the payments depend on the survival of a particular person. Life annuities are discussed in Chapter 3.

Consider an annuity certain consisting of n payments of \$1 each, the first payment being due 1 year from now, the second due 2 years from now, and so on. The last (nth) payment will be due n years from now. The present value of this series of payments is the sum of the present values of the individual payments. We saw in section 12 that the present value of \$1 due in n years is v^n. Therefore the present value of this annuity is $v + v^2 + v^3 + \cdots + v^n$. The standard symbol for the present value of an annuity certain immediate is $a_{\overline{n}|}$. Accordingly, we can write

$$a_{\overline{n}|} = v + v^2 + v^3 + \cdots + v^n. \tag{4}$$

(The word "immediate" implies that the first annual payment is made at the *end* of the first year. An annuity in which the first payment is made at the *beginning* of the first year is called an annuity due.)

The present value of an annuity certain due which is to run for n years is denoted by $\ddot{a}_{\overline{n}|}$. Under such an annuity the first payment is due right now, the second 1 year from now, the third 2 years from now, and so on. The last payment is due $n - 1$ years from now. Since the present value of the annuity is the sum of the present values of the individual payments, we have

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-1}. \tag{5}$$

From equation 4 we have the equation

$$a_{\overline{n-1}|} = v + v^2 + v^3 + \cdots + v^{n-1},$$

and thus obtain immediately the valuable relation

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}. \tag{6}$$

This relation allows us to compute values of $\ddot{a}_{\overline{n}|}$ painlessly if we have tables of $a_{\overline{n}|}$. Values of $a_{\overline{n}|}$ at 2½% are given in Table 1 in Appendix 3.

The reader will recall from algebra that a geometric progression is a series of terms in which the ratio of each term (except the first) to the term just preceding is constant. For example, the series 1, 2, 4, 8, 16 represents a geometric progression whose first term is 1, whose ratio is 2, and whose number of terms is 5.

Formula 4 can be written in more compact form by observing that the right-hand side of the equation is a geometric progression with a ratio v. Applying the familiar formula [sum $= a(1 - r^n)/(1 - r)$, where a is the first term, r the ratio, and n the number of terms] for the sum of the terms of a geometric progression, we have

$$a_{\overline{n}|} = \frac{v(1 - v^n)}{1 - v} = \frac{1 - v^n}{\frac{1}{v} - 1} = \frac{1 - v^n}{1 + i - 1} = \frac{1 - v^n}{i} \tag{7}$$

since

$$v = \frac{1}{1 + i}.$$

Similarly,

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{iv}. \tag{8}$$

It may be that we are interested in knowing how much the annuity payments accumulated at compound interest will amount to in n years. $s_{\overline{n}|}$ represents the sum which the payments under the annuity immediate will amount to n years from now. The accumulated amount of the series of payments is the sum of the accumulated amounts of the individual payments. The first payment, made at the end of the first year, will accumulate at interest for $n - 1$ years and amount to $(1 + i)^{n-1}$ at the end of the nth year. Similarly, the second payment will amount to $(1 + i)^{n-2}$, and so on. Therefore,

$$s_{\overline{n}|} = (1 + i)^{n-1} + (1 + i)^{n-2} + \cdots + (1 + i) + 1. \tag{9}$$

Since the right-hand side is a geometric progression where

$$a = (1 + i)^{n-1}, \ r = v, \ \text{and} \ n = n,$$

we have

$$s_{\overline{n}|} = \frac{(1+i)^{n-1}(1-v^n)}{1-v}$$

$$= \frac{(1+i)^{n-1}(1-v^n)(1+i)}{(1-v)(1+i)}$$

$$= \frac{(1+i)^n - 1}{1+i-1} \quad [\text{since } v(1+i) = 1]$$

or

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}. \tag{10}$$

Similarly, $\ddot{s}_{\overline{n}|}$ represents the amount n years from now under an annuity due, and

$$\ddot{s}_{\overline{n}|} = (1+i)^n + (1+i)^{n-1} + \cdots + (1+i)^2 + (1+i) \tag{11}$$

or

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{iv}. \tag{12}$$

It follows from formulas 9 and 11 that

$$\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1. \tag{13}$$

Values of $s_{\overline{n}|}$ are given in Table 1.

EXAMPLE 1

A certain life insurance policy provides that, when Mr. Jones dies, Mrs. Jones has the option of receiving $10,000 in cash or receiving a series of equal payments at the beginning of each year for 10 years. Find the annual payment.

Solution: Let R be the annual payment. Then the present value of the series of 10 payments is represented by $R \cdot \ddot{a}_{\overline{10}|}$. But this present value must be equal to $10,000. Therefore, we have the equation

$$R\ddot{a}_{\overline{10}|} = 10{,}000,$$

and

$$R = \frac{10{,}000}{\ddot{a}_{\overline{10}|}}$$

$$= \frac{10{,}000}{1 + a_{\overline{9}|}}$$

$$= \frac{10{,}000}{8.970866}$$

$$= \$1114.72.$$

(The value of $a_{\overline{9}|}$ is found in Table 1.)

EXAMPLE 2

As shown above in section 12, the rate of discount d may be expressed as iv. Prove that

$$d = 1 - v.$$

Solution:

$$\begin{aligned} d &= iv \\ &= \frac{i}{1+i} \\ &= \frac{1+i-1}{1+i} \\ &= \frac{1+i}{1+i} - \frac{1}{1+i} \\ &= 1 - v. \end{aligned}$$

EXAMPLE 3

Find the present value of an annuity certain of \$1000 per year if the first payment is due 10 years from now and the last 20 years from now.

Solution: There are apparently 11 payments. One year before the first payment is due (i.e., 9 years from now) the value of the annuity is $1000a_{\overline{11}|}$.* Since $1000a_{\overline{11}|}$ 9 years from now is the same as $v^9 1000a_{\overline{11}|}$ now, we have for the present value of this annuity

$$\begin{aligned} 1000v^9 a_{\overline{11}|} &= (1000)(0.8007284)(9.514209) \\ &= \$7618.30. \end{aligned}$$

Another way of working this problem, which saves some arithmetic, is to observe that the required present value

$$\begin{aligned} &= 1000a_{\overline{20}|} - 1000a_{\overline{9}|} \\ &= 15{,}589.162 - 7970.866 \\ &= \$7618.30. \end{aligned}$$

PROBLEM SET 7

1. How much will \$5000 accumulated at $2\frac{1}{2}$% interest amount to in 5 years? 10 years? 40 years?

* Strictly speaking, this should be written $\$1000a_{\overline{11}|}$ rather than $1000a_{\overline{11}|}$ (without the dollar sign). Using a dollar sign every time extreme precision demands tends to clutter up the page. In this book precision in such matters will be sacrificed for readability.

2. What is the present value of \$1000 due in 10 years? 20 years? 50 years?

3. Fifty dollars is paid into a fund at the beginning of every year for 20 years. How much will there be in the fund at the end of 20 years? just after the last payment has been made?

4. What equal deposits must be placed in a fund at the end of every year for 15 years in order to have \$10,000 in the fund at the end of 25 years?

5. Find the present value of an annuity certain immediate of \$100 annually to run for 27 years.

6. Find the present value of an annuity certain of \$50 annually if the first payment is due in 5 years and the last in 30 years.

7. Find the present value of an annuity certain under which 10 annual payments of \$300 followed by 20 annual payments of \$200 are to be made with the first payment due immediately.

8. In lieu of taking the \$10,000 face amount of a life insurance policy, the beneficiary desires to receive \$1000 at the start of every year for 10 years and a final payment at the end of 10 years. Compute the amount of the final payment.

9. A beneficiary under an old policy which guarantees $3\frac{1}{2}$ % interest on funds left with the company elects to receive the interest at the end of every year for 20 years on the policy proceeds of \$10,000 and to receive the principal sum of \$10,000 at the end of 20 years. If money is worth $2\frac{1}{2}$ %, what is the present value of the payments under this mode of settlement?

10. Prove the following identities:

(a) $v^n + v^{n+1} + v^{n+2} + \cdots + v^{n+m} = a_{\overline{n+m}|} - a_{\overline{n-1}|}$.

(b) $\ddot{a}_{\overline{n}|} = \dfrac{1 - v^n}{d}$.

(c) $\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{n-1}|} + v^n$.

(d) $i\ddot{a}_{\overline{n}|} = 1 + i - v^{n-1}$.

(e) $d = \dfrac{1}{\ddot{a}_{\overline{n}|}} - \dfrac{1}{\ddot{s}_{\overline{n}|}}$.

11. A house is bought for \$5000 cash and payments of \$600 at the end of every year for 15 years. What is the equivalent cash price if money is worth $2\frac{1}{2}$ %?

12. A house is bought for \$8000 cash and payments of \$400 at the end of every year for 15 years. What is the equivalent cash price if money is worth $2\frac{1}{2}$ %?

13. A \$1000 bond matures July 1, 1960. Interest of \$50 is payable every July 1. Compute the value of the bond just after the 1950 interest payment has been made if money is worth $2\frac{1}{2}$ %.

14. A corporation purchases a machine for \$10,000. A fund is set up in order to have enough money on hand to replace this machine when necessary. The following assumptions are made:

(a) The machine will last 10 years.
(b) The price of a new machine will be the same 10 years from now as it is now.
(c) Payments will be made into the fund at the end of every year.
(d) The fund will earn $2\frac{1}{2}$ % interest.

How much money should be paid into the fund every year?

C H A P T E R   T H R E E

Life Annuities

14 · INTRODUCTION

In the last section an annuity certain was defined as "a series of periodic payments payable over a fixed period of time, it being assumed that each payment is certain to be made when it is due." A life annuity may be defined as a series of periodic payments where each payment will actually be made only if a designated life is alive at the time the payment is due. There are various types of life annuities, some of which we shall now define. Other types will be discussed later in this chapter and in Chapter 6. In this section we shall assume that payments are always made annually.

A *whole-life annuity immediate* is a series of payments under which the first payment is due 1 year from now, the second 2 years from now, and so on, with each payment being contingent on the survival of a designated life. If the amount of each annual payment is \$1 and the designated life is age x, the present value of the annuity is represented by the symbol a_x. A *whole-life annuity due* is a series of payments under which the first payment is due at once, the second 1 year from now, the third 2 years from now, and so on, with each payment being contingent on the survival of a designated life. If the amount of each annual payment is \$1 and the designated life is age x, the present value of the annuity is represented by the symbol $\ddot{a}_x$. The reader will note from the definitions that the only difference between the two types is the payment of \$1 due right now. Since the present value of this payment is \$1, we have the very useful relation

$$\ddot{a}_x = 1 + a_x. \tag{14}$$

The symbol $\ddot{a}_{x:\overline{n}|}$ represents the present value of a series of annual payments of \$1 continuing for not more than n payments; the first payment is due right now, and each payment is contingent upon the survival of (x). The annuity ceases after n payments have been made

even if (x) is still alive. Such an annuity is called a *temporary life annuity due.*

The symbol $a_{x:\overline{n}|}$ represents the present value of a similar temporary annuity with the first payment due a year from now. Such an annuity is called a *temporary life annuity immediate.* The reader should note that the only difference between an annuity due for n years and an annuity immediate for $n - 1$ years is the payment of \$1 right now. Hence we have the useful relation

$$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}. \tag{15}$$

The symbol ${}_{n|}\ddot{a}_x$ represents the present value of a series of annual payments of \$1 starting when (x) attains age $x + n$ and continuing so long as (x) shall be alive. Such an annuity is called a *deferred life annuity due* or a life annuity due on (x) deferred n years.

The symbol ${}_{n|}a_x$ represents the present value of a deferred life annuity immediate for \$1 a year. Payments begin when (x) attains age $x + n + 1$ and continue so long as (x) shall be alive. There is no difference between a life annuity immediate deferred for n years and a life annuity due deferred for $n + 1$ years. That is,

$${}_{n+1|}\ddot{a}_x = {}_{n|}a_x \tag{16}$$

The difference in the form of the two sides of equation 16 involves the fundamental concepts of annuity due and annuity immediate. An annuity due is payable at the *beginning* of eacb year, whereas an annuity immediate is payable at the *end* of each year. Tbe end of one year and the beginning of the next represent the same point in time. Referring to formula 16, ${}_{n+1|}\ddot{a}_x$ represents the present value of an annuity which will be "entered upon" at the end of $n + 1$ years with a payment being due at the beginning of each year that (x) enters; ${}_{n|}a_x$ represents the present value of an annuity wbich will be entered upon at the end of n years but with payments made at the end of every year that (x) enters *and* completes. In effect, an annuity immediate always involves a 1-year deferment in addition to the deferment represented by the number of years to elapse until the annuity is entered upon.

The symbol ${}_{n|t}\ddot{a}_x$ represents the present value of a *deferred temporary life annuity due* for \$1 a year. The first payment is due when (x) attains age $x + n$; payments continue for t years, the last being due when (x) attains age $x + n + t - 1$. Each payment is contingent on the survival of (x).

The symbol ${}_{n|t}a_x$ represents the present value of a *deferred temporary life annuity immediate* for \$1 a year, in which the first payment is

due when (x) attains age $x + n + 1$ and the last at age $x + n + t$. From these definitions of deferred temporary life annuities we have

$$_{n+1|t}\ddot{a}_x = {}_{n|t}a_x. \tag{17}$$

Note: Whenever the words "life annuity" are used without qualification, a whole-life annuity is implied.

15 · WHOLE-LIFE ANNUITY

Suppose that l_x persons, all aged x, contribute enough money into a fund so that \$1 can be paid from the fund to each person who survives to age $x + 1$, another dollar to each person who survives to age $x + 2$, and so on. Payments are to be made until all original persons are dead. The problem is to find how much money each of the persons must contribute to the fund. Let this amount be Z. Then the total amount in the fund at the start must be l_xZ, which must equal the present value of all the payments to be made from the fund.

Out of the l_x persons now age x, l_{x+1} will survive to receive \$1 a year from now. That is to say that l_{x+1} dollars will be paid from the fund 1 year from now. The present value of that disbursement is vl_{x+1}. Similarly, the present value of the payments to be made 2 years from now is v^2l_{x+2}, etc. Hence, we have the equation

$$l_xZ = vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \cdots + v^{\omega-x}l_\omega.$$

Now the amount Z which each person contributes to the fund is apparently the net single premium (or present value) for a whole-life annuity immediate. According to section 14 this present value is denoted by the symbol a_x. Substituting a_x for Z and solving, we have

$$a_x = \frac{vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \cdots + v^{\omega-x}l_\omega}{l_x}. \tag{18}$$

This method of derivation is commonly called the "mutual fund method." The name "mutual fund" comes from the assumption that a number of imaginary people contribute to a fund whose proceeds are used to pay benefits to the contributors (or to their beneficiaries in the case of life insurance plans) according to terms mutually agreed upon. The fund may be regarded as existing for the mutual benefit of the participants.

If we assume mortality according to the CSO table and interest at 2½%, we can calculate the present value of a whole-life annuity

immediate for any age by substituting in equation 18 and performing the necessary arithmetic. For example,

$$a_{20} = \frac{(1.025)^{-1}l_{21} + (1.025)^{-2}l_{22} + (1.025)^{-3}l_{23} + \cdots + (1.025)^{-79}l_{99}}{l_{20}}$$

$$= \frac{(1.025)^{-1}(949{,}171) + (1.025)^{-2}(946{,}789) + \cdots + (1.025)^{-79}(125)}{951{,}483},$$

which can be calculated to any desired accuracy. The only catch is that the arithmetic is very onerous. In order to save arithmetic a system of six symbols called commutation symbols has been developed. These six symbols and their definitions are as follows:

$$\left.\begin{aligned} D_x &= v^x l_x; \\ N_x &= D_x + D_{x+1} + D_{x+2} + \cdots + D_\omega; \\ S_x &= N_x + N_{x+1} + N_{x+2} + \cdots + N_\omega; \\ C_x &= v^{x+1} d_x; \\ M_x &= C_x + C_{x+1} + C_{x+2} + \cdots + C_\omega; \\ R_x &= M_x + M_{x+1} + M_{x+2} + \cdots + M_\omega. \end{aligned}\right\} \quad (19)$$

Given a rate of interest and a mortality table, numerical values of these commutation symbols can be calculated for all values of x. These numerical values are customarily presented in tabular form and are called commutation columns. CSO 2½% commutation columns are shown in Table 3.

The practical value of D_x and N_x in simplifying calculations and formulas will be shown now. The value of the other symbols will be demonstrated in later sections.

If we multiply both numerator and denominator of the right-hand side of formula 18 by v^x, we have

$$a_x = \frac{v^{x+1}l_{x+1} + v^{x+2}l_{x+2} + v^{x+3}l_{x+3} + \cdots + v^\omega l_\omega}{v^x l_x}.$$

Since $v^x l_x = D_x$, $v^{x+1}l_{x+1} = D_{x+1}$, etc., we have

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \cdots + D_\omega}{D_x},$$

and, since $N_{x+1} = D_{x+1} + D_{x+2} + D_{x+3} + \cdots + D_\omega$, by definition, we have

$$a_x = \frac{N_{x+1}}{D_x}. \tag{20}$$

Thus, instead of performing the rather terrifying computing job called for by formula 18, we can calculate a_x for any age by looking up two numbers in the commutation columns and performing one division.

Equation 18 can be derived by another method, which uses the concept of mathematical expectation. a_x is the sum of the present values of a series of \$1 payments to be made at the end of every year, provided that (x) is alive when the payment is due. The present value of any payment is the product of (a) the present value of the payment on the assumption that the payment will certainly be made and (b) the probability that it will be made—i.e., the mathematical expectation of the person who is scheduled to receive the payment. Using this concept, we have

$$\begin{aligned} a_x &= vp_x + v^2 \cdot {}_2p_x + v^3 \cdot {}_3p_x + \cdots + v^{\omega-x} \cdot {}_{\omega-x}p_x \\ &= v\frac{l_{x+1}}{l_x} + v^2\frac{l_{x+2}}{l_x} + v^3\frac{l_{x+3}}{l_x} + \cdots + v^{\omega-x}\frac{l_\omega}{l_x} \\ &= \frac{vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \cdots + v^{\omega-x}l_\omega}{l_x} \\ &= \frac{N_{x+1}}{D_x}. \end{aligned}$$

This method of derivation might be called the "discount method."

Note that the only reason for stopping at age ω in this and in similar series is that all terms beyond ω are 0. Theoretically, the series is infinite.

From formula 14 we have

$$\ddot{a}_x = 1 + a_x = 1 + \frac{N_{x+1}}{D_x} = \frac{D_x + N_{x+1}}{D_x}$$

or

$$\ddot{a}_x = \frac{N_x}{D_x} \tag{21}$$

since

$$D_x + N_{x+1} = D_x + (D_{x+1} + D_{x+2} + D_{x+3} + \cdots + D_\omega) = N_x.$$

Values of $\ddot{a}_x$, $1000/\ddot{a}_x$, and $1000/a_x$ are given in Table 4.

$1/\ddot{a}_x$ and $1/a_x$ are frequently written $\ddot{a}_x^{-1}$ and a_x^{-1}, respectively. Thus, $1000/\ddot{a}_x$ could be written $1000\ddot{a}_x^{-1}$.

EXAMPLE 1

Compute the net single premium for a whole-life annuity immediate of $1200 per year purchased (*a*) at age 80, (*b*) at age 18.

Solution:

$$
\begin{aligned}
\text{(a) Net single premium} &= 1200a_{80} \\
&= 1200(\ddot{a}_{80} - 1) \\
&= (1200)(4.102345) \quad \text{from Table 4} \\
&= \$4922.81 \quad \text{to the nearest cent.} \\
\text{(b) Net single premium} &= 1200a_{18} \\
&= 1200\,\frac{N_{19}}{D_{18}} \\
&= \frac{(1200)(16{,}340{,}808)}{612{,}917.42} \\
&= \$31{,}992.84.
\end{aligned}
$$

The reason for using commutation columns in part *b* is that Table 4 doesn't go below age 20.

EXAMPLE 2

Prove that $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$.

Solution:

$$
\begin{aligned}
1 + vp_x\ddot{a}_{x+1} &= 1 + \frac{vl_{x+1}}{l_x}\,\ddot{a}_{x+1} \\
&= 1 + \frac{v^{x+1}l_{x+1}}{v^x l_x}\,\ddot{a}_{x+1} \\
&= 1 + \frac{D_{x+1}}{D_x}\cdot\frac{N_{x+1}}{D_{x+1}} \\
&= 1 + \frac{N_{x+1}}{D_x} \\
&= \frac{D_x + N_{x+1}}{D_x} \\
&= \frac{N_x}{D_x} \\
&= \ddot{a}_x.
\end{aligned}
$$

EXAMPLE 3

Compute the net single premium for example 1*a*, assuming 1937 Standard Annuity mortality.

Solution: If the annuitant is male,

$$\begin{aligned}\text{Net single premium} &= 1200a_{80} \\ &= 1200(5.76148) \\ &= \$6913.78.\end{aligned}$$

If the annuitant is female,

$$\begin{aligned}\text{Net single premium} &= 1200a_{75} \\ &= 1200(7.34361) \\ &= \$8812.33.\end{aligned}$$

PROBLEM SET 8

1. Compute the net single premium for a whole-life annuity immediate of \$2000 per year purchased at age (*a*) 75, (*b*) 15.

2. A man aged 60 has \$20,000 cash. What annual annuity payment can he purchase if the first payment is at age (*a*) 60, (*b*) 61?

3. A life aged 20 agrees to pay an insurance premium of \$100 at the beginning of each year as long as he lives. What single payment at age 20 is equivalent to the annuity represented by these insurance premiums?

4. An individual aged 25 pays \$5000 to a life insurance company. The company agrees that (*a*) if the individual should die before age 65, the company shall pay to his heirs the \$5000 plus interest accumulated to the date of his death; (*b*) if he lives to age 65, the company will pay to him a certain amount at the beginning of each year as long as he lives. Find the annual payment beginning at age 65.

5. An individual aged 25 agrees to pay \$200 to an insurance company at the beginning of every year for 40 years or until prior death. If he should die before making the fortieth payment, his estate will receive the accumulated value of his payments. If he lives to make the fortieth payment, his original contract will be exchanged for a contract under which the company agrees to pay him *R* dollars at the end of every year as long as he lives. Calculate *R*.

6. A 22-year-old girl has been left an estate of \$100,000, which is invested at 4%. The will provides that she is to receive the income as long as she lives. Find the present value of her inheritance.

7. Prove the following identities:

$$(a)\quad \ddot{a}_{x+1} = \frac{(1+i)a_x}{p_x}.$$

$$(b)\quad a_x = vp_x + v^2 \cdot {}_2p_x \cdot \ddot{a}_{x+2}.$$

8. Prove that $a_x < 1/i$.

9. Prove that $(1+i)a_x < e_x$.

10. Check the value of D_{30} in Table 3.

11. Prove that $N_{20} - N_{25} = D_{20} + D_{21} + D_{22} + D_{23} + D_{24}$.

12. Prove that $D_x + 2D_{x+1} + 3D_{x+2} = S_x - S_{x+3} - 3N_{x+3}$.

13. A whole-life annuity immediate to (x) provides for a payment of (1.025) at the end of the first year, $(1.025)^2$ at the end of the second year, $(1.025)^3$ at the end of the third year, etc. Show that the present value of this annuity is e_x.

14. According to the American Experience Table at $3\frac{1}{2}\%$,

$$a_{20} = 20.144, \; a_{21} = 20.013, \; a_{22} = 19.878, \; l_{22} = 91192.$$

Find d_{20}.

Hint: See the identity in example 2 of section 15.

15. Work problem 1 on the assumption of 1937 Standard Annuity mortality for (*a*) males; (*b*) females.

16. Work problem 2 on the assumption of 1937 Standard Annuity mortality.

17. Work problem 16, replacing "man" with "woman."

18. Work problem 3 on the assumption of 1937 Standard Annuity mortality for (*a*) males; (*b*) females.

16 · TEMPORARY AND DEFERRED LIFE ANNUITIES

The following formulas can be derived by either the mutual fund method or the discount method. The actual derivation is suggested as an exercise for the reader.

$$a_{x:\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x}. \tag{22}$$

$$\ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}. \tag{23}$$

$$_{n|}a_x = \frac{N_{x+n+1}}{D_x}. \tag{24}$$

$$_{n|}\ddot{a}_x = \frac{N_{x+n}}{D_x}. \tag{25}$$

$$_{n|t}a_x = \frac{N_{x+n+1} - N_{x+n+t+1}}{D_x}. \tag{26}$$

$$_{n|t}\ddot{a}_x = \frac{N_{x+n} - N_{x+n+t}}{D_x}. \tag{27}$$

Table 8 shows values of $\ddot{a}_{x:\overline{n}|}$; Table 9 shows values of $1000/\ddot{a}_{x:\overline{n}|}$.

Example 1

Prove algebraically and verbally that

$$\ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x = \ddot{a}_x.$$

Solution:

$$\ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x = \frac{N_x - N_{x+n}}{D_x} + \frac{N_{x+n}}{D_x}$$

$$= \frac{N_x}{D_x}$$

$$= \ddot{a}_x.$$

A verbal interpretation is as follows: $\ddot{a}_{x:\overline{n}|}$ is the present value of a temporary life annuity due for n years, first payment at age x and last at age $x + n - 1$; that is, $\ddot{a}_{x:\overline{n}|}$ is the present value of the first n payments under a life annuity due beginning at age x. ${}_{n|}\ddot{a}_x$ is the present value of a deferred life annuity, first payment at age $x + n$; that is, ${}_{n|}\ddot{a}_x$ is the present value of all the payments, except the first n, under a life annuity due beginning at age x. Together, $\ddot{a}_{x:\overline{n}|}$ and ${}_{n|}\ddot{a}_x$ represent the present value of all payments under a life annuity due at age x, since the payments represented by ${}_{n|}\ddot{a}_x$ begin where those represented by $\ddot{a}_{x:\overline{n}|}$ leave off. Since $\ddot{a}_x$ is the present value of a life annuity due to (x), we have

$$\ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x = \ddot{a}_x.$$

Example 2

A man now aged 45 has \$10,000 cash. What annual annuity payment, beginning at age 65, can he purchase?

Solution: Let R be the annual annuity payment. Then, equating present values,

$$R \cdot {}_{20|}\ddot{a}_{45} = 10{,}000$$

and

$$R = \frac{10{,}000}{{}_{20|}\ddot{a}_{45}}$$

$$= \frac{10{,}000}{N_{65}/D_{45}}$$

$$= \frac{10{,}000 D_{45}}{N_{65}}$$

$$= \frac{(10{,}000)(280{,}638.95)}{1{,}172{,}129.8}$$

$$= \$2394.27.$$

Alternatively,

$$
\begin{aligned}
{}_{20|}\ddot{a}_{45} &= \ddot{a}_{45} - \ddot{a}_{45:\overline{20|}} \\
&= 18.393726 - 14.217079 \\
&= 4.176647
\end{aligned}
$$

so that

$$
\begin{aligned}
R &= \frac{10{,}000}{4.176647} \\
&= \$2394.27.
\end{aligned}
$$

In this particular problem the alternative solution saves little, if any, arithmetic. In many problems, however, the relation ${}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n|}}$ does save a noticeable amount of arithmetic and is a useful relation to remember.

Example 3

Solve the problem of example 2 on the assumption of 1937 Standard Annuity mortality.

Solution:

$$
\begin{aligned}
R &= \frac{10{,}000D_{45}}{N_{65}} \\
&= \frac{(10{,}000)(300{,}940)}{1{,}653{,}095} \\
&= \$1820.46.
\end{aligned}
$$

Example 4

Solve the problem of example 2 on the assumption that mortality follows the 1937 Standard Annuity Table (set back 1 year).

Solution: The effect of a 1-year setback is to assume that the man is 1 year younger than he actually is. Thus, we assume in this case that the man is 44 years old and proceed from there.

$$
\begin{aligned}
R \cdot {}_{20|}\ddot{a}_{44} &= 10{,}000; \\
R &= \frac{10{,}000D_{44}}{N_{64}} \\
&= \frac{(10{,}000)(310{,}293)}{1{,}798{,}004} \\
&= \$1725.76.
\end{aligned}
$$

EXAMPLE 5

A woman now aged 45 has \$10,000 cash. What annual annuity payment, beginning at age 65, can she purchase? Assume 1937 Standard Annuity mortality.

Solution: The 1937 Standard Annuity Table considers that a woman aged 45 is equivalent to a man aged 40. Thus,

$$R \cdot {}_{20|}\ddot{a}_{40} = 10{,}000;$$

$$R = \frac{10{,}000 D_{40}}{N_{60}}$$

$$= \frac{(10{,}000)(349{,}299)}{2{,}452{,}392}$$

$$= \$1424.32.$$

EXAMPLE 6

Solve the problem of example 5 on the assumption that mortality follows the 1937 Standard Annuity Table (set back 2 years).

Solution: In this example the total setback is 7 years, 5 for sex and 2 for the setback in the table. Thus,

$$R \cdot {}_{20|}\ddot{a}_{38} = 10{,}000;$$

$$R = \frac{10{,}000 D_{38}}{N_{58}}$$

$$= \frac{(10{,}000)(369{,}853)}{2{,}825{,}590}$$

$$= \$1308.94.$$

PROBLEM SET 9

1. Derive formula 22 by the mutual fund method.
2. Derive formula 27 by the discount method.
3. Prove formula 15 by using commutation symbols.
4. Find the net single premium for the following annuities paying \$100 a year issued to a person aged 55:
   (*a*) Temporary life annuity due to run for 10 years.
   (*b*) Temporary life annuity immediate to run for 10 years.
   (*c*) Temporary life annuity immediate to run for 9 years.
   (*d*) Whole-life annuity due, deferred 20 years.
   (*e*) Whole-life annuity immediate, deferred 10 years.
   (*f*) Temporary life annuity due to run 10 years, deferred 15 years.
   (*g*) Temporary life annuity immediate to run 10 years, deferred 15 years.
   (*h*) Whole-life annuity, first payment at age 60.

5. Find the present value of a whole-life annuity, deferred 39 years, for $1000 annually issued to (21).

6. A man now aged 32 has $15,000 cash. What annual annuity payment, beginning at age 60, can he purchase?

7. A man aged 40 wishes to make equal payments at the beginning of each year for 25 years in order to pay off a debt of $8000. If his remaining debt is to be canceled at his death, compute the annual payment.

8. Solve problem 7 on the assumption that the unpaid balance of the debt is not canceled at death.

9. A man aged 33 purchases a life insurance policy calling for an annual premium of $200 payable at the beginning of each year for 20 years or until prior death. Find the net single premium equivalent to this series of payments.

10. A man aged 50 wishes to exchange $20,000 cash for a temporary life annuity to run for 20 years with the first payment to be made immediately. Find the annual payment under the annuity.

11. A man aged 26 agrees to pay to a life insurance company $2000 at the beginning of every year for 20 years or until prior death. In return the life insurance company agrees to pay a sum of money every year as long as the man is alive with the first payment due at age 60. Find the amount of the annual payment to be made by the company. *Hint:* Present value of the man's payments = present value of company's payments.

12. Prove the inequalities:

(a) $a_{x:\overline{n|}} < a_{\overline{n|}}$.

(b) ${}_{n|t}\ddot{a}_x < v^n \ddot{a}_{\overline{t|}}$.

13. Complete the following table, assuming that $i = 1/9$ and carrying all calculations to 3 decimals.

x	l_x	d_x	q_x	p_x	a_x	e_x	$\mathring{e}_x$
95	1000	200					
96							
97	500			0.600			
98							
99	100						
100	0						

14. Find the net single premium for a whole-life annuity, deferred to age 70, for $1000 annually issued to a man aged 60 under the following mortality assumptions:

(a) CSO.
(b) 1937 Standard Annuity.
(c) 1937 Standard Annuity (set back 1 year).
(d) 1937 Standard Annuity (set back 2 years).

15. Solve problem 14 if "man" is changed to "woman."

17 · PURE ENDOWMENT

A pure endowment is a payment to be made at the end of a specified number of years if a certain life survives to the end of the period. The symbol used to denote the present value of an n-year pure endowment

on the life of a person aged x is ${}_nE_x$. Thus, the present value of a 20-year pure endowment on (30) would be ${}_{20}E_{30}$.

Using the discount method explained in section 15, we have

$$\begin{aligned}{}_nE_x &= v^n \cdot {}_np_x \\ &= v^n \frac{l_{x+n}}{l_x} \\ &= \frac{v^{x+n}l_{x+n}}{v^x l_x}\end{aligned}$$

or

$${}_nE_x = \frac{D_{x+n}}{D_x}. \tag{28}$$

It may be noted that a pure endowment is a special case of a life annuity; it is a deferred temporary life annuity where $n = 1$. Logically, the symbol ${}_{n|1}\ddot{a}_x$ could be used instead of ${}_nE_x$. However, ${}_{n|1}\ddot{a}_x$ is not used in practice.

It may also be noted that an annuity may be regarded as a series of pure endowments. Thus,

$$\begin{aligned}a_{x:\overline{n|}} &= {}_1E_x + {}_2E_x + {}_3E_x + \cdots + {}_nE_x \\ &= \frac{D_{x+1}}{D_x} + \frac{D_{x+2}}{D_x} + \frac{D_{x+3}}{D_x} + \cdots + \frac{D_{x+n}}{D_x} \\ &= \frac{N_{x+1} - N_{x+n+1}}{D_x}.\end{aligned}$$

Values of $1000{}_nE_x$ and $1/{}_nE_x$ are shown in Tables 6 and 7, respectively.

Example 1

A man now aged 20 is promised \$10,000 when he reaches age 40. Find the present value of the promise.

Solution:

$$\begin{aligned}\text{Present value} &= 10{,}000 \cdot {}_{20}E_{20} \\ &= (10)(566.56606) \quad \text{from Table 6} \\ &= \$5665.66.\end{aligned}$$

EXAMPLE 2

Prove algebraically that

$$_{n|}\ddot{a}_x = {}_nE_x \cdot \ddot{a}_{x+n}.$$

Solution:

$$_nE_x \cdot \ddot{a}_{x+n} = \frac{D_{x+n}}{D_x} \cdot \frac{N_{x+n}}{D_{x+n}}$$

$$= \frac{N_{x+n}}{D_x}$$

$$= {}_{n|}\ddot{a}_x.$$

PROBLEM SET 10

1. Find the present value of a \$1000 pure endowment payable in 20 years to a person now aged (*a*) 20, (*b*) 30, (*c*) 50, and (*d*) 70.

2. A man now aged x has \$1000 cash. If he deposits this with an insurance company, what amount should he receive when he attains age 65 if x is equal to (*a*) 57, (*b*) 50, (*c*) 41, (*d*) 36, (*e*) 29, (*f*) 20, and (*g*) 19?

3. How much will \$1000 invested at 2½ % compound interest amount to in 45 years?

4. Two payments of \$10,000 each are to be received at the end of 5 and 10 years, respectively. Find their present value:
(*a*) If they are certain to be received.
(*b*) If they are to be received only if (20) is alive to receive them.
(*c*) If they are to be received only if (60) is alive to receive them.

5. Prove the following identities:

$$(a)\quad D_{x+1} = vp_xD_x.$$

$$(b)\quad \ddot{a}_{x:\overline{m+n|}} = \ddot{a}_{x:\overline{m|}} + {}_mE_x \cdot \ddot{a}_{x+m:\overline{n|}}.$$

$$(c)\quad a_x = a_{x:\overline{n|}} + {}_nE_x \cdot a_{x+n}.$$

$$(d)\quad {}_mE_x \cdot {}_tE_{x+m} = {}_{m+t}E_x.$$

6. Use identity 5*d* to evaluate $_9E_{25}$ without using commutation columns.

7. Compute the numerical value of the product

$$_1E_{40} \cdot {}_1E_{41} \cdot {}_1E_{42} \cdot \cdot \cdot {}_1E_{57}.$$

8. Find the present value of a promise to pay (22) \$1000 when and if he attains an age equal to 22 plus the curtate expectation of life of (22).

9. Find the present value of a \$1000 pure endowment payable in 20 years to a man aged 45 under the following mortality assumptions:
(*a*) CSO.
(*b*) 1937 Standard Annuity.
(*c*) 1937 Standard Annuity (set back 1 year).
(*d*) 1937 Standard Annuity (set back 2 years).

10. Solve problem 9 if "man" is changed to "woman."

18 · FOREBORNE ANNUITY

Suppose that l_x individuals, all aged x, enter into an agreement whereby each individual contributes now $1 to a fund; a year from now each of the l_{x+1} survivors contributes another dollar to the fund; 2 years from now each of the l_{x+2} survivors contributes still another dollar, and so on until contributions have been collected n times; n years from now the amount in the fund will be divided equally among the l_{x+n} survivors. Let ${}_nu_x$ represent the share of each survivor, and assume that the fund has been invested at compound interest at rate i. Then

$$
\begin{aligned}
{}_nu_x &= \frac{l_x(1+i)^n + l_{x+1}(1+i)^{n-1} + l_{x+2}(1+i)^{n-2} + \cdots + l_{x+n-1}(1+i)}{l_{x+n}} \\
&= \frac{v^{x+n}l_x(1+i)^n + v^{x+n}l_{x+1}(1+i)^{n-1} + \cdots + v^{x+n}l_{x+n-1}(1+i)}{v^{x+n}l_{x+n}} \\
&= \frac{v^x l_x + v^{x+1}l_{x+1} + \cdots + v^{x+n-1}l_{x+n-1}}{v^{x+n}l_{x+n}} \\
&= \frac{D_x + D_{x+1} + \cdots + D_{x+n-1}}{D_{x+n}}
\end{aligned}
$$

or

$$
{}_nu_x = \frac{N_x - N_{x+n}}{D_{x+n}}. \tag{29}
$$

This type of fund is called a *tontine fund*, and each survivor's share is called a foreborne annuity. Tontines were invented by an Italian named Tonti (1630–1695). Under the first French State tontine of 1689, the State, in return for a loan of 25 million livres, paid 1¼ million livres yearly in interest. The contributors were divided into ten groups, with each group receiving 125,000 livres yearly. Since this annual payment was shared by the survivors, the last survivor in each group would receive the entire 125,000 livres annually until his death. There have been other tontines conducted by various states as well as by private enterprise, but the public has not looked with favor on schemes of this kind, with the result that restrictive legislation has been adopted.

If n happens to be 1, the 1 is usually omitted. Thus

$$
u_x = \frac{N_x - N_{x+1}}{D_{x+1}}
$$

or

$$u_x = \frac{D_x}{D_{x+1}}. \tag{30}$$

The symbol ${}_nu_x$ is frequently called the value of the individual survivor's payments accumulated with the benefit of interest and survivorship.

Table 11 gives values of ${}_nu_x$.

To the authors' knowledge, neither the pure endowment nor the foreborne annuity are currently issued in the United States as individual policies by any commercial life insurance company. The concepts are, however, extremely useful in life insurance calculations.

19. GENERAL ANNUITY FORMULA.

$$R \cdot \frac{N_x - N_{x+n}}{D_y} \tag{31}$$

represents the value at age y of a life annuity with annual payments equal to R, the first payment to be made at age x and the number of payments to be limited to a maximum of n.

The reader can satisfy himself as to the validity of this formula by studying the other formulas in this chapter or by producing an algebraic proof.

PROBLEM SET 11

1. Solve problem 11, set 9, by equating the value of the man's payments to the value of the company's payments as of age 46.

2. Find the present value of a promise to pay to (34) $100 every year for 26 years, followed by $200 every year for 5 years, first payment to be made immediately.

3. A man aged 30 is to receive $10,000 if he is alive at age 40. He wishes to exchange this privilege for a life annuity beginning at age 65. Express in commutation symbols the amount he should receive every year under the life annuity.

4. A young man of 20 has been left an estate of $100,000 which is invested at 5%. The will provides that if he lives he is to receive the income annually for the next 10 years and the principal of the estate when he reaches age 30. Find the present value of his inheritance.

5. A $100,000 policy provides that 3½% interest will be paid on all death proceeds left with the company. The policy contains a settlement agreement providing that the wife of the insured will receive the interest on the $100,000 at the end of each year as long as she livee. At her death the $100,000 is to be paid in cash to the daughter of the insured. If the wife is 54 years old at the death of her husbsnd, what is the present vslue of her shsre of the insursnce proceeds?

6. A $100,000 policy provides that 3½% interest will be paid on all death proceeds left with the company. The policy contains a settlement agreement providing that the wife of the insured will receive the interest on the $100,000 at

the end of every year for 10 years and that at the end of the 10 years she will receive the principal sum, provided she is alive at the time each payment is due. If the wife is 54 years old at the death of her husband, what is the present value to her of the payments to be made under the settlement agreement?

7. A large number of persons all aged 23 enter into an agreement whereby each person contributes R dollars to a fund at the beginning of each year, provided he survives. The last payment is due at age 64. At age 65 the amount in the fund will be divided equally among the survivors, and each survivor's share will be used to purchase a life annuity paying \$3000 at the beginning of every year. Calculate R.

8. A person aged 23 wishes to pay for a life annuity of \$3000 annually, first payment at age 65, by means of 42 equal annual payments, the first payment to be made immediately. If no refund is to be made in the event of death, what is the annual payment?

9. Prove the identities:

$$(a)\quad \ddot{a}_{x:\overline{n}|} = {}_nE_x \cdot {}_nu_x.$$

$$(b)\quad {}_{m+n}u_x = \frac{{}_mu_x}{{}_nE_{x+m}} + {}_nu_{x+m}.$$

$$(c)\quad \ddot{a}_{x+1} = u_x a_x.$$

$$(d)\quad {}_{10|10}\ddot{a}_x = \frac{{}_{10}u_{x+10} \cdot {}_{30|}\ddot{a}_x}{{}_{10|}\ddot{a}_{x+20}}.$$

10. Prove algebraically and by general reasoning that ${}_nu_x > s_{\overline{n}|}$.

PROBLEM SET 12

1. Compute the net single premium for a whole-life annuity due of \$3000 per year purchased at age 55.

2. Compute the net single premium for a temporary life annuity immediate to run for 20 years, issued to (45) and paying \$1000 per year.

3. Find the present value of a \$2000 pure endowment payable in 35 years to a person now aged 25.

4. A person now aged 24 purchases a pure endowment payable at age 65. The purchase price is \$1000. Find the amount of the pure endowment.

5. Mr. Hagen loans Mr. Jones a certain sum of money. The loan is to be repaid by 20 annual payments of \$1000 each, the first payment due 1 year after the date of the loan. Death of Mr. Jones does not cancel the loan. Find the amount of the loan.

6. Compare the results of problems 2 and 5. Discuss the difference.

7. A man aged 50 purchases from an insurance company a whole life annuity paying \$200 at the end of every month. The man is killed in an automobile accident $5\frac{1}{2}$ months after the date of the contract. What equitable payment should the insurance company make to the man's estate? Discuss.

8. Find the present value of a promise to pay (33) \$100 every year for 27 years, followed by \$200 every year for 5 years, first payment to be made immediately.

9. Find the present value of a promise to pay (33) \$200 every year for 20 years, followed by \$100 every year for 7 years, first payment to be made immediately.

10. Prove that $\ddot{a}_{\overline{n}|} > \ddot{a}_{x-n:\overline{n}|}$.

C H A P T E R   F O U R

Life Insurance

20 · INTRODUCTION

The essential feature of an insurance plan, whether it be life insurance or something else, is the cooperation of a large number of individuals who agree, in effect, to share the cost of the individual losses that may occur. In life insurance this cooperation is usually attained through the instrument of a life insurance company which insures a large number of individuals. The company is able to take advantage of the law of averages and charge premiums so calculated that the cost of the death claims of the individuals who die will be borne by all the individuals insured. It is the ability to spread the losses among a great many individuals plus the ability to handle the necessary detail efficiently and to invest funds safely at a reasonable rate of interest that makes the life insurance business sound.

When an individual is insured by a company, he and the company agree to a written contract drafted by the company and approved by regulatory authority. This contract is called a policy. The policy provides that the policyholder will make certain payments, called gross premiums, to the company and that the company will pay a certain amount of money, called the face amount, if certain events occur. The person to whom the face amount is payable in the event of death is called the beneficiary. The date of issue is the date on which the policy becomes effective. The year beginning with the date of issue and ending 1 year after the date of issue is called the first policy year. The second policy year begins when the first ends, and so on. The age at issue is usually the age nearest birthday; that is, the exact age of the insured in years taken to the nearest integer.

In the present chapter our attention will be confined to net premiums. Net premium calculations assume that mortality and interest will follow certain assumptions, that there are no expenses in connection with the insurance, and that all death claims are paid at the end of the policy year in which death occurs. Some of the problems

in connection with gross premiums will be considered in Chapters 7, 8, and 9.

SETTLEMENT OPTIONS. The policy usually provides that on the death of the insured the beneficiary may elect an optional mode of settlement in lieu of taking the face amount in cash. Typical options include:

(*a*) The right to leave the face amount with the insurance company and to receive interest on this amount at a rate specified in the policy. The amount left with the company under this option may ordinarily be withdrawn on demand. Partial withdrawal may or may not be allowed.

(*b*) An annuity certain for a specified period of time.

(*c*) A whole-life annuity.

(*d*) A whole-life annuity with the payments for the first 10 years guaranteed, whether or not the original beneficiary dies during the 10-year period.

A life insurance company may allow a beneficiary to combine two or more options. Thus, the beneficiary under a $20,000 policy may be allowed to place $10,000 under option *a* and the other $10,000 under option *c*.

Such options are ordinarily referred to as settlement options. The payments under each option are usually specified in the policy and are calculated so that their present value (as of date of death) is equal to the face amount of the policy.

Since this book is primarily concerned with principles and methods of calculation rather than with actuarial judgment, problems involving settlement options will be calculated on the assumption of CSO mortality and $2\frac{1}{2}\%$ interest, unless otherwise specified. At the present time life insurance companies are generally basing their settlement options on more conservative assumptions so that the payments are somewhat less than they would be under CSO $2\frac{1}{2}\%$. There are two major reasons for using these more conservative assumptions: (*a*) Annuitant mortality is lighter than CSO mortality. (*b*) Settlement options are guaranteed at the time the policy is issued. Since a number of years will usually elapse until the death of the insured, guaranteeing a settlement option is a very long-range proposition. In the meantime annuitant mortality may improve further and the interest rate may drop.

In order to emphasize the financial difference between basing settlement options on an insurance table and on an annuity table, several settlement option problems will be based on the 1937 Standard Annuity Table at $2\frac{1}{2}\%$ interest.

21 · TERM INSURANCE

Under a term insurance policy the sum insured hecomes payahle provided the person insured dies within a stated period, called the term of the policy. The simplest form of term insurance is 1-year term, which will be considered first.

Suppose that l_x persons, all aged x, agree that each will contrihute an amount Z to a fund and that at the end of a year \$1 will he paid to the beneficiary of every one of the l_x persons who died during the year. The contrihutions, together with 1-year's interest on them, is supposed to be exactly enough to pay the death claims, leaving nothing in the fund. We have

$$Z \cdot l_x(1 + i) = d_x$$

since the total fund at the start of the year is Zl_x and the numher of dollars to be paid at the end of the year is equal to the numher of deaths during the year; namely, d_x. Hence,

$$Z = \frac{d_x}{l_x(1 + i)}$$

$$= \frac{vd_x}{l_x}.$$

Z is apparently the net premium for 1-year term insurance at age x. The symhol for this premium, sometimes called the natural premium, is c_x. We have

$$c_x = \frac{vd_x}{l_x}$$

$$= \frac{v^{x+1}d_x}{v^x l_x}$$

or

$$c_x = \frac{C_x}{D_x} \tag{32}$$

from the definition of C_x in equations 19.

The expression for c_x could also be derived by the discount method. c_x is the product of (a) the value of a payment of \$1 due at the end of the year on the assumption that it is certain to he paid and (b) the prohability that it will he paid. That is,

$$c_x = vq_x = v\frac{d_x}{l_x} = \frac{C_x}{D_x}.$$

Now let $A^{1}_{x:\overline{n|}}$ denote the present value of term insurance for \$1 on (x) to run for n years; that is, the present value of a promise to pay \$1 at the end of the year in which (x) dies, provided the death occurs before age $x + n$. Using the discount method of analysis, we have

$$\begin{aligned} A^{1}_{x:\overline{n|}} &= v\frac{d_x}{l_x} + v^2\frac{d_{x+1}}{l_x} + v^3\frac{d_{x+2}}{l_x} + \cdots + v^n\frac{d_{x+n-1}}{l_x} \\ &= \frac{v^{x+1}d_x + v^{x+2}d_{x+1} + v^{x+3}d_{x+2} + \cdots + v^{x+n}d_{x+n-1}}{v^x l_x} \\ &= \frac{C_x + C_{x+1} + C_{x+2} + \cdots + C_{x+n-1}}{D_x} \end{aligned}$$

or

$$A^{1}_{x:\overline{n|}} = \frac{M_x - M_{x+n}}{D_x}, \tag{33}$$

since

$$M_x = C_x + C_{x+1} + C_{x+2} + \cdots + C_{x+n-1} + C_{x+n} + C_{x+n+1} + \cdots + C_\omega$$

and

$$M_{x+n} = C_{x+n} + C_{x+n+1} + \cdots + C_\omega$$

by the definitions in equations 19.

Values of $1000c_x$ and $1000A^{1}_{x:\overline{n|}}$ are shown in Tables 5 and 9, respectively. c_x and $A^{1}_{x:\overline{1|}}$ are, of course, the same thing.

Example

Prove algebraically and verbally that

$$A^{1}_{x:\overline{n|}} = c_x + {}_1E_x c_{x+1} + {}_2E_x c_{x+2} + \cdots + {}_{n-1}E_x c_{x+n-1}.$$

Solution:

$$\begin{aligned} c_x &+ {}_1E_x c_{x+1} + {}_2E_x c_{x+2} + \cdots + {}_{n-1}E_x c_{x+n-1} \\ &= \frac{C_x}{D_x} + \frac{D_{x+1}}{D_x}\cdot\frac{C_{x+1}}{D_{x+1}} + \frac{D_{x+2}}{D_x}\cdot\frac{C_{x+2}}{D_{x+2}} + \cdots + \frac{D_{x+n-1}}{D_x}\cdot\frac{C_{x+n-1}}{D_{x+n-1}} \\ &= \frac{C_x + C_{x+1} + C_{x+2} + \cdots + C_{x+n-1}}{D_x} \\ &= \frac{M_x - M_{x+n}}{D_x} \\ &= A^{1}_{x:\overline{n|}}. \end{aligned}$$

c_x is the value at age x of \$1's worth of term insurance covering the period between ages x and $x + 1$. c_{x+1} is the value at age $x + 1$ of \$1's worth of term insurance covering the period between ages $x + 1$ and $x + 2$. Since \$1 at age $x + 1$ is equivalent to ${}_1E_x$ at age x, discounting at both interest and mortality, c_{x+1} at age $x + 1$ must be equivalent to ${}_1E_x c_{x+1}$ at age x. That is to say that ${}_1E_x c_{x+1}$ represents the value at age x of \$1's worth of term insurance covering the period between ages $x + 1$ and $x + 2$; similarly for the other terms on the right-hand side of the equation. In particular, the last term, ${}_{n-1}E_x c_{x+n-1}$ is the value at age x of \$1's worth of term insurance covering the period between ages $x + n - 1$ and $x + n$. Collectively, therefore, the series of terms on the right-hand side represents the value at age x of \$1's worth of term insurance for each of the n years between ages x and $x + n$. But this value is $A^{1}_{x:\overline{n}|}$ by definition of the symbol. Therefore

$$A^{1}_{x:\overline{n}|} = c_x + {}_1E_x c_{x+1} + {}_2E_x c_{x+2} + \cdots + {}_{n-1}E_x c_{x+n-1}.$$

PROBLEM SET 13

1. Derive formula 33 by the mutual fund method.

2. Calculate the net single premium for a \$1000 20-year term insurance policy issued at age (*a*) 15, (*b*) 20, (*c*) 30, (*d*) 40, (*e*) 50, (*f*) 60, (*g*) 70.

3. Calculate the net single premium for a \$10,000 term insurance policy issued to (31) to run for (*a*) 1 year, (*b*) 3 years, (*c*) 10 years, (*d*) 20 years, (*e*) 30 years, (*f*) 34 years.

4. A man aged 35 pays a net single premium of \$500 for 10-year term insurance. Find (to the nearest dollar) the face amount of the policy.

5. Find the net single premium required for a man aged 55 to purchase \$5000 of term insurance to age 70 plus a life annuity of \$600 per year, first payment to be made at age 70.

6. Find the net single premium required for a man aged 37 to purchase \$5000 of 20-year term insurance plus a \$5000 20-year pure endowment.

7. (*a*) Find the net single premium for a \$1000 term to age 65 policy issued at age 59.

(*b*) Find the sum of the natural premiums for the six ages 59 to 64.

(*c*) Why are the answers to (*a*) and (*b*) different?

8. A policy is issued to a man aged 40 providing for \$10,000 of insurance for 10 years followed by \$5000 for 15 years. Find the net single premium.

9. A policy is issued to a man aged 40 providing for \$2000 of insurance for 2 years followed by \$5000 of insurance for 18 years. Find the net single premium.

10. Prove algebraically and verbally that

$$A^{1}_{x:\overline{20}|} = A^{1}_{x:\overline{10}|} + {}_{10}E_x \cdot A^{\;1}_{x+10:\overline{10}|}.$$

11. The death benefit under a life insurance policy is \$20,000. The policy provides that, at the death of the insured, the beneficiary may elect one of the following options in lieu of the \$20,000 cash.

(*a*) A 20-year annuity certain due.

(*b*) A whole-life annuity due.

(*c*) A whole-life annuity due with the provision that the first 10 payments will be made whether the original beneficiary is alive to receive them or not.

Compute the annual payment under each option if the beneficiary is 50 years old at the date of death of the insured.

Note: Option *c* may be regarded as the sum of an annuity certain and a deferred life annuity.

12. Solve problem 11 *c* if the beneficiary is a man under the following mortality assumptions:

(*a*) 1937 Standard Annuity.

(*b*) 1937 Standard Annuity (set back 1 year).

(*c*) 1937 Standard Annuity (set back 2 years).

13. Solve problem 12 if the beneficiary is a woman.

22 • WHOLE-LIFE INSURANCE

Term insurance, because of its comparative cheapness, is the best insurance buy for some situations. It has, however, distinct disadvantages. When the term period elapses, there is no more insurance although the need for insurance may not have disappeared. Many policies eliminate this objection by providing that an individual insured under a term policy may renew the policy at the end of the term without medical examination if application to renew is made a reasonable length of time before the term expires. The only difficulty with this arrangement is that the probability of death in a year increases with age so that the premium increases at each renewal, becoming prohibitive at the older ages.

Whole-life insurance is a device by which an individual can have a policy which will pay the face amount at his death no matter when death occurs and which requires a constant (level) premium. The premiums may be paid every year for the life of the insured, or they may be limited to a maximum number of payments, such as 20. The problem of annual premiums will be discussed in detail in section 24. In this section we shall confine ourselves to the question of the net single premium, or present value, for whole-life insurance.

A_x represents the present value of \$1 of whole-life insurance on the life of (x); that is, A_x is the present value of a promise to pay \$1 at the end of the year in which (x) dies. Using the discount method of analysis, we have

$$\begin{aligned} A_x &= v\frac{d_x}{l_x} + v^2\frac{d_{x+1}}{l_x} + v^3\frac{d_{x+2}}{l_x} + \cdots + v^{\omega-x+1}\frac{d_\omega}{l_x} \\ &= \frac{v^{x+1}d_x + v^{x+2}d_{x+1} + v^{x+3}d_{x+2} + \cdots + v^{\omega+1}d_\omega}{v^x l_x} \\ &= \frac{C_x + C_{x+1} + C_{x+2} + \cdots + C_\omega}{D_x} \end{aligned}$$

or

$$A_x = \frac{M_x}{D_x}. \tag{34}$$

Table 4 gives numerical values for $1000A_x$ and $1/A_x$.

EXAMPLE

Find the net single premium for a life insurance policy providing a \$1000 death benefit for 5 years and \$2000 thereafter if the age at issue is 30.

Solution: The net single premium may be considered as the difference between the net single premium for \$2000 of whole-life insurance and \$1000 of 5-year term. That is, the required net single premium

$$\begin{aligned} &= 2000A_{30} - 1000A^{\,1}_{30:\overline{5}|} \\ &= 2(413.80049) - 18.10263 \\ &= \$809.50. \end{aligned}$$

PROBLEM SET 14

1. Calculate the net single premium for a \$1000 whole-life insurance policy issued at age (*a*) 19, (*b*) 20, (*c*) 47.

2. Calculate the face amount of whole-life insurance purchased by a single premium of \$1000 at age (*a*) 19, (*b*) 20, (*c*) 47.

3. Find the net single premium for a life insurance policy providing a \$15,000 death benefit for 10 years and \$10,000 thereafter if the age at issue is 36.

4. Find the net single premium for a life insurance policy providing a \$10,000 death benefit for 10 years and \$15,000 thereafter if the age at issue is 36.

5. A man aged 39 has \$10,000 cash. He wishes to apply this amount to the purchase of a single premium policy which provides a death benefit of \$30,000 for 5 years followed by a constant death benefit for the rest of his life, the amount to be whatever his single premium will purchase. Find the death benefit after the first 5-year period has elapsed.

6. Calculate the net single premium at age 75 for:

(*a*) A \$1000 whole-life policy.

(*b*) A \$1000 20-year term policy.

7. Prove the following identities:

(*a*) $A_x = v(q_x + p_x A_{x+1}).$

(*b*) $vq_x = \dfrac{A_x - vA_{x+1}}{1 - A_{x+1}}.$

(*c*) $A_x u_x = A_{x+1} + \dfrac{q_x}{p_x}.$

(*d*) $A_{x+10} \cdot {}_{10}E_x = A_x - A^{\,1}_{x:\overline{10}|}.$

8. Prove algebraically and by general reasoning that

$$A_x = c_x + {}_1E_x \cdot c_{x+1} + {}_2E_x \cdot c_{x+2} + \cdots.$$

23 · ENDOWMENT INSURANCE

Endowment insurance provides for the payment of (*a*) the face amount as a death benefit if the insured dies during a certain period of years or (*b*) the face amount as an endowment benefit if the insured survives to the end of the period. Mathematically, endowment insurance is the sum of term insurance and pure endowment. We have the relation

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x \tag{35}$$

where $A_{x:\overline{n}|}$ represents the present value of an n-year endowment insurance for $1. From equation 35,

$$\begin{aligned} A_{x:\overline{n}|} &= A^{1}_{x:\overline{n}|} + {}_nE_x \\ &= \frac{M_x - M_{x+n}}{D_x} + \frac{D_{x+n}}{D_x} \end{aligned}$$

or

$$A_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \tag{36}$$

EXAMPLE

Find to the nearest cent the net single premium for a $1000 20-year endowment insurance policy issued (*a*) at age 20, and (*b*) at age 15.

Solution:

$$\begin{aligned} (a)\ \text{Net single premium} &= 1000A_{20:\overline{20}|} \\ &= 1000A^{1}_{20:\overline{20}|} + 1000{}_{20}E_{20} \\ &= 53.89923 + 566.56606 \\ &= \$620.47. \end{aligned}$$

$$\begin{aligned} (b)\ \text{Net single premium} &= 1000A_{15:\overline{20}|} \\ &= 1000\,\frac{M_{15} - M_{35} + D_{35}}{D_{15}} \\ &= (1000)\,\frac{203570.08 - 174423.84 + 381995.63}{664414.29} \\ &= (1000)\,\frac{411141.87}{664414.29} \\ &= \$618.80. \end{aligned}$$

The reason for using commutation columns in working part *b* is that the tables in this book for $A^{1}_{x:\overline{n}|}$ and ${}_nE_x$ don't happen to include age 15.

PROBLEM SET 15

1. Derive formula 35 by the discount method.

2. Compute the net single premium for a 20-year endowment insurance policy issued for \$1000 at age (*a*) 15, (*b*) 25, (*c*) 26, (*d*) 43, (*e*) 57, (*f*) 70, (*g*) 80.

3. Compare the answer for problem 2*g* with $1000A_{80}$ and explain the relation.

4. Compute the net single premium for a \$5000 endowment insurance policy issued at age 25 to run for (*a*) 3 years, (*b*) 7 years, (*c*) 10 years, (*d*) 30 years, (*e*) 40 years, (*f*) 60 years.

5. A man aged 50 has \$10,000 cash to apply as a single premium for 10-year endowment insurance. Find the face amount of the policy.

6. A man aged 50 has \$10,000 cash to apply as a single premium for 11-year endowment insurance. Find the face amount of the policy.

7. A life insurance policy issued to (21) provides for a death benefit of \$3000 in the event of death before age 60. If the policyholder survives to age 60, the sum of \$5000 is paid to him. Calculate the net single premium for this policy.

8. A life insurance policy issued to (35) provides for a death benefit of \$1000 in the event of death before age 70. If the policyholder survives to age 70, the sum of \$500 is paid to him. Calculate the net single premium for this policy.

9. A man aged 30 pays \$800 for a policy which provides for a death benefit of \$1000 in the event of death before age 65 plus a payment of \$*R* in the event of survival to age 65. Find *R*.

10. A policy issued to a person aged 40 provides for a death benefit of \$1000 in the event of death within 25 years. In the event of survival to age 65, the policy is to be exchanged for a contract which provides a life annuity of \$100 per year, the first payment to be made at age 65. Calculate the net single premium.

11. Solve problem 10 if the first 10 annuity payments are guaranteed under the annuity contract whether the annuitant lives or dies.

12. Describe in words the benefits whose net single premium is described by:

(*a*) $A_{40:\overline{10|}}$.

(*b*) $$\frac{M_{30} - M_{70}}{D_{30}}.$$

(*c*) $$\frac{1000(M_{20} - M_{75}) + 500D_{75}}{D_{20}}.$$

(*d*) $1000A^{\,1}_{30:\overline{40|}} + 100 \cdot {}_{40}E_{30} + 50 \cdot {}_{40|}a_{30}$.

13. Solve problem 10 if the insured is a male on the assumption of CSO mortality to age 65 and 1937 Standard Annuity mortality after age 65.

14. Solve problem 13 if the insured is a female.

24 · ANNUAL PREMIUMS

Although some insurance is sold on a single-premium basis, the usual procedure is to have premiums payable at the beginning of every year, or some shorter period such as a month. The problems involved in making premium payments oftener than once a year (i. e., fractionally) will be considered in Chapter 6. Whether the premiums are payable

annually or fractionally, the premiums do not usually change from year to year. We can think of the insurance as being paid for by means of equal annual (or fractional) installments.

Life insurance policies are given names that tend to be descriptive of both the benefits and the method of paying premiums. However, these names do not distinguish between annual and fractional premiums.

Policies under which the benefit is *whole-life* insurance are usually classified in three ways. *Ordinary life* requires that an annual premium be paid for life. *Limited-payment life* requires that an annual premium be paid until the insured's death or until a certain number of premiums have been paid, whichever comes first. The premiums are limited to a certain maximum number. A whole-life policy issued to a person aged 25 providing for a maximum of 40 equal annual premiums is referred to as either a "40-pay life" policy or a "paid-up at age 65" policy. *Single-premium life* requires just one premium. It is, of course, a special case of limited-payment life. The fewer the premium payments, the larger the premium. The table below shows the net premiums for various types of whole life policies issued at age 20. As usual, we are assuming CSO mortality and $2\frac{1}{2}\%$ interest.

Plan of Insurance	*Net Premium for \$1000*
Ordinary life	\$ 12.49
Paid-up at 85	12.52
Paid-up at 65	13.50
Paid-up at 60	14.17
30-pay life	16.53
25-pay life	18.58
20-pay life	21.76
15-pay life	27.19
10-pay life	38.19
Single-premium life	338.68

"Limited-payment" life insurance is very frequently referred to in the insurance business as "limited-pay" life insurance. In this book the words "payment" and "pay" are used interchangeably in this connection.

The number of annual premiums called for by term insurance policies is almost always the same as the number of years for which the insurance is in effect. Thus 5-year term provides term insurance for 5 years and requires 5 equal annual premiums. Because the probability of death increases with age, the longer the term, the higher the premium. The table below shows CSO $2\frac{1}{2}\%$ net premiums for age at issue 20.

Plan of Insurance	*Net Premium for $1000*
5-year term	$2.53
10-year term	2.77
15-year term	3.07
20-year term	3.46
Term to 60	6.35
Term to 65	7.49

Endowment insurance can be classified into three major types. *n-year endowment* (or endowment at age $x + n$) provides endowment insurance for n years and requires n annual premiums. *Limited-pay endowment* requires that premiums be paid for a limited period which is less than the term of the endowment. *Single-premium endowment* requires just one premium. The table below shows CSO 2½% net premiums for age at issue 20.

Note: Since the word "endowment," standing by itself, could conceivably refer to either pure endowment or endowment insurance, the reader may be wondering how to tell which is meant. In this book the word "endowment," unqualified, will always mean endowment insurance. The word "pure" will never be omitted in referring to a pure endowment. This is consistent with common usage in the insurance business in the United States.

Plan of Insurance	*Net Premium for $1000*
Endowment at 85	$ 12.58
Endowment at 65	15.46
Endowment at 60	17.45
30-year endowment	24.43
25-year endowment	30.48
20-year endowment	39.87
15-year endowment	55.88
10-year endowment	88.36
20-pay endowment at 85	21.87
20-pay endowment at 65	24.93
Single-premium 20-year endowment	620.47
Single-premium endowment at 65	387.91

Note: Many companies and life insurance men consider an "endowment at 85" as a whole-life plan. Economically it's not much different. Actually, of course, a whole-life plan is really an endowment at 100 on the CSO table. If a person with a whole-life policy should live to be 100, the face amount of the policy would be paid as an endowment. The American Experience Table assumed that no one would reach age 96. Therefore, a whole-life policy issued on the basis of that table is really an endowment at 96. This explains the appearance of an occasional newspaper story in which the statement is made that

John Doe has collected on his insurance by reaching age 96 and outliving the mortality table.

The rate of interest used in the calculations has a very definite effect on the net premium. The table below shows the net premiums for various plans at four different rates of interest. CSO mortality is assumed throughout. The age at issue is 20. It should be observed

	Net Premium for $1000			
Plan of Insurance	*2%*	*2¼%*	*2½%*	*3%*
Ordinary life	$ 13.86	$ 13.15	$ 12.49	$ 11.29
20-pay life	25.51	23.54	21.76	18.70
10-pay life	45.72	41.74	38.19	32.15
Single-premium life	414.08	374.11	338.68	279.26
10-year term	2.79	2.78	2.77	2.75
Endowment at 65	17.05	16.23	15.46	14.02
20-year endowment	41.99	40.92	39.87	37.85

that, the lower the rate of interest, the higher the net premiums.

A number of symbols to denote annual premiums of different kinds are in common use. Some of these symbols are shown below.

P_x is the net annual premium for $1 of ordinary life insurance issued at age x.

${}_nP_x$ is the net annual premium for $1 of n-pay life insurance issued at age x.

$P^{1}_{x:\overline{n|}}$ is the net annual premium for $1 of n-year term insurance issued at age x.

$P_{x:\overline{n|}}$ is the net annual premium for $1 of n-year endowment insurance issued at age x.

${}_tP_{x:\overline{n|}}$ is the net annual premium for $1 of t-pay n-year endowment insurance issued at age x.

It is not recommended that these annual premium symbols be used in working out an actual problem. In most, if not all, problems the best method of attack is to let P be the unknown premium and solve the fundamental equation for P. The fundamental equation is

Present value of future premiums = Present value of future benefits.

Example 1

Without referring to Table 13, calculate the net annual premium for a $2000 ordinary life policy issued at age 30.

Solution: Let P be the net annual premium. Then, equating the present value of future premiums to the present value of future benefits, we have

$$P\ddot{a}_{30} = 2000A_{30},$$

$$P = \frac{2000A_{30}}{\ddot{a}_{30}}$$

$$= \frac{2(413.80049)}{24.034180}$$

$$= \$34.43.$$

EXAMPLE 2

Find the net annual premium for $10,000 of 20-pay life issued at age 19.

Solution: Let P be the net annual premium. Then

$$P\ddot{a}_{19:\overline{20}|} = 10000A_{19}.$$

Since $\ddot{a}_{19:\overline{20}|}$ is not in our tables, we have to use commutation symbols. The above equation becomes

$$P\frac{N_{19} - N_{39}}{D_{19}} = 10000\frac{M_{19}}{D_{19}},$$

$$P = \frac{10000M_{19}}{N_{19} - N_{39}}$$

$$= \frac{1980363800}{9293057}$$

$$= \$213.10.$$

EXAMPLE 3

Find the net annual premium for $1000 of 20-pay 30-year endowment issued at age 25.

Solution:

$$P\ddot{a}_{25:\overline{20}|} = 1000A_{25:\overline{30}|},$$

$$P = \$32.81.$$

PROBLEM SET 16

1. Calculate the net annual premium for a $1000 life insurance policy issued at age 30 for the following plans:

(a) Ordinary life.
(b) Paid-up at 70.
(c) Paid-up at 55.
(d) 15-pay life.
(e) Term to 70.
(f) Endowment at 65.
(g) 5-year endowment.
(h) 20-pay, endowment at 65.
(i) 1-year term.

2. Express in commutation symbols the net annual premium for a $1000 policy issued at age 45 for the following plans:

(a) Ordinary life.
(b) 30-pay life.
(c) 10-year term.
(d) Endowment at 85.
(e) 20-pay, endowment at 68.

3. Express in commutation symbols

$$(a)\ P_{50},\ (b)\ {}_{20}P_{15},\ (c)\ P^{1}_{40:\overline{10|}},\ (d)\ P_{40:\overline{10|}},\ (e)\ {}_{10}P_{20:\overline{40|}}.$$

4. Compute the net annual premium for an 18-pay endowment at 85 issued to a person aged 42 for $2000 face amount.

5. A person aged 27 has a 20-year endowment policy which was taken out 3 years ago. The net annual premium is $100. What is the face amount of the policy?

6. Express in commutation symbols the net annual premium for a whole-life insurance policy to (x) which provides for a death benefit of $1000 at the end of the year of death of the insured and $1000 every year thereafter for 9 years.

7. Calculate the net annual premium for a $5000 per year whole-life annuity to begin at age 70 if the age at issue is 30 and premiums are limited to 20.

8. A man aged 30 agrees to pay a lump sum of $2000 and an annual premium (the first one due a year from now) of $100 for a whole-life policy. Find the face amount of the policy.

9. Prove that

$$P_x = \frac{vq_x + P_{x+1}a_x}{\ddot{a}_x}.$$

10. If $q_x = 0.01x$ and $v = 0.9$, calculate P_{98}.

25 • VARIOUS ALGEBRAIC RELATIONS

Various fundamental relations involving the commutation symbols and the net premium symbols exist. These relations are of practical importance in many instances. Knowledge of the relations combined with reasonable ingenuity can frequently save calculation time or be used to apply independent checks in connection with actuarial calculations.

$$\begin{aligned} C_x &= v^{x+1}d_x \quad \text{(by definition)} \\ &= v^{x+1}(l_x - l_{x+1}) \\ &= v(v^x l_x) - v^{x+1}l_{x+1} \end{aligned}$$

or

$$C_x = vD_x - D_{x+1}. \tag{37}$$

Since equation 37 is a perfectly general relation, holding for all values of x, we have

$$C_{x+1} = vD_{x+1} - D_{x+2},$$

$$C_{x+2} = vD_{x+2} - D_{x+3},$$

$$\cdots$$

$$C_{\omega-1} = vD_{\omega-1} - D_{\omega},$$

$$C_{\omega} = vD_{\omega} - 0 \quad (\text{since } D_{\omega+1} = 0).$$

Adding these equations, we have

$$\begin{aligned} C_x + C_{x+1} + C_{x+2} + \cdots + C_{\omega} \\ = v(D_x + D_{x+1} + D_{x+2} + \cdots + D_{\omega}) \\ - (D_{x+1} + D_{x+2} + D_{x+3} + \cdots + D_{\omega}) \end{aligned}$$

or

$$M_x = vN_x - N_{x+1}. \tag{37a}$$

Dividing both sides of equation 37a by D_x gives

$$A_x = v\ddot{a}_x - a_x. \tag{38}$$

Using the relations, $d = 1 - v$ and $a_x = \ddot{a}_x - 1$, and substituting in equation 38, we have

$$A_x = (1 - d)\ddot{a}_x - (\ddot{a}_x - 1)$$

$$= \ddot{a}_x - d\ddot{a}_x - \ddot{a}_x + 1$$

or

$$A_x = 1 - d\ddot{a}_x. \tag{39}$$

Dividing both sides of equation 39 by $\ddot{a}_x$ and noting that $A_x/\ddot{a}_x = P_x$, we have

$$P_x = \frac{1 - d\ddot{a}_x}{\ddot{a}_x} = \frac{1}{\ddot{a}_x} - d. \tag{40}$$

EXAMPLE 1

Prove algebraically and verbally that

$$A^{1}_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}.$$

Solution:

$$v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} = v\,\frac{N_x - N_{x+n}}{D_x} - \frac{N_{x+1} - N_{x+n+1}}{D_x}$$

$$= \frac{(vN_x - N_{x+1}) - (vN_{x+n} - N_{x+n+1})}{D_x}$$

$$= \frac{M_x - M_{x+n}}{D_x} \quad \text{by equation } 37a$$

$$= A^{1}_{x:\overline{n}|}.$$

The verbal interpretation involves rather nice reasoning. It goes as follows:

$v\ddot{a}_{x:\overline{n}|}$ is the present value of a series of annual payments, each equal to v, continuing for not more than n payments; the first payment is due right now, and each payment is contingent upon the survival of (x). In other words, $v\ddot{a}_{x:\overline{n}|}$ is the present value of a series of payments of v to be made at the beginning of each year that (x) enters, with a maximum of n payments. Since a payment of v at the *beginning* of each year that (x) *enters* is equivalent to a payment of 1 at the *end* of each year that (x) *enters*, we can describe $v\ddot{a}_{x:\overline{n}|}$ as the present value of a series of payments of 1 to be made at the end of each year that (x) enters but subject to a maximum of n payments.

$a_{x:\overline{n}|}$ is the present value of a series of payments of 1 to be made at the end of each year that (x) *completes*.

The only difference between the payments under $v\ddot{a}_{x:\overline{n}|}$ and under $a_{x:\overline{n}|}$ is that, if (x) dies before age $x + n$, a payment will be made at the end of the year of death under $v\ddot{a}_{x:\overline{n}|}$ because (x) will have entered that year, but a payment will not be made at the end of the year of death under $a_{x:\overline{n}|}$ because (x) will not have completed the year. Therefore, on subtracting $a_{x:\overline{n}|}$ from $v\ddot{a}_{x:\overline{n}|}$ we have the present value of a payment of 1 to be made at the end of the year of death of (x), provided death occurs before age $x + n$; i.e., $A^{1}_{x:\overline{n}|}$. Hence,

$$A^{1}_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}.$$

EXAMPLE 2

Give a verbal proof of equation 39.

Solution: One dollar invested at a rate of interest i will pay iv or d at the beginning of every year, and the principal of \$1 will be available intact at the end of every year. In particular, \$1 so invested will provide a life annuity due to (x) of d per year and still leave a dollar

to be returned at the end of the year in which (x) dies. Equating present values, we have

$$1 = d\ddot{a}_x + A_x$$

or

$$A_x = 1 - d\ddot{a}_x.$$

PROBLEM SET 17

1. Prove that:

(a) $R_x = vS_x - S_{x+1}$.

(b) $M_x = D_x - dN_x$.

(c) $A_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$.

(d) $P_x = P_{x-1} + \dfrac{P_x - c_{x-1}}{\ddot{a}_{x-1}}$.

(e) $P^{1}_{x:\overline{n}|} = v - \dfrac{a_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$.

(f) $A_x = v - da_x$.

(g) $\dfrac{A_x - A_y}{1 - A_y} + \dfrac{\ddot{a}_x}{\ddot{a}_y} = \dfrac{P_r \ddot{a}_r}{A_r}$.

2. Interpret problem 1c verbally.

3. Complete the following table, expressing each of the symbols $\ddot{a}_x$, A_x, and P_x entirely in terms of each of the others and the discount factor d.

	$\ddot{a}_x$	A_x	P_x
$\ddot{a}_x$	$\ddot{a}_x$		
A_x	$1 - d\ddot{a}_x$	A_x	
P_x	$\dfrac{1}{\ddot{a}_x} - d$		P_x

4. If $A_x = 0.01x$ and $i = 0.02$, find expressions for $\ddot{a}_x$ and P_x.

26 · DEFERRED INSURANCE

The net single premium for an n-year term insurance of \$1, deferred r years, is represented by the symbol ${}_{r|}A^{1}_{x:\overline{n}|}$. To say it another way, ${}_{r|}A^{1}_{x:\overline{n}|}$ is the present value of a promise to pay \$1 at the end of the year in which (x) dies, provided that he dies between ages $x + r$ and $x + r + n$.

Similarly, ${}_{r|}A_x$ is the net single premium for a whole-life insurance of \$1 deferred r years, and ${}_{r|}A_{x:\overline{n}|}$ is the net single premium for an n-year endowment insurance deferred r years.

These net single premiums are expressed in terms of commutation

symbols in equations 41. The derivation of these equations is suggested as an exercise for the student.

$$\left.\begin{aligned} {}_{r|}A_x &= \frac{M_{x+r}}{D_x}; \\ {}_{r|}A^{\,1}_{x:\overline{n|}} &= \frac{M_{x+r} - M_{x+r+n}}{D_x}; \\ {}_{r|}A_{x:\overline{n|}} &= \frac{M_{x+r} - M_{x+r+n} + D_{x+r+n}}{D_x}. \end{aligned}\right\} \qquad (41)$$

27 • ACCUMULATED COST OF INSURANCE

The accumulated cost of insurance is frequently referred to as the cost of insurance—omitting the word "accumulated." It is strictly an actuarial concept and should not be confused with "net cost," an expression dear to the hearts of many agents. Net cost, as the term is used in nontechnical insurance conversation, may mean several things.

Perhaps the simplest way to explain the more common popular meanings of net cost is by using the following formulas:

(*a*) "History average net cost" $= \dfrac{G - D}{n}$;

(*b*) "History average surrender net cost" $= \dfrac{G - D - S}{n}$;

(*c*) "Current dividend scale average net cost" $= \dfrac{G' - D'}{n}$;

(*d*) "Current dividend scale average surrender net cost" $= \dfrac{G' - D' - S'}{n}$;

where G is the sum of the gross premiums paid during the first n policy years under a policy which was issued n years ago.

D is the sum of the cash dividends paid during the first n policy years under a policy which was issued n years ago.

S is the cash surrender value available at the end of n years (i.e., now) under a policy which was issued n years ago.

G' is the sum of the gross premiums to be paid during the first n policy years under a policy issued currently.

D' is the sum of the cash dividends to be paid during the first n policy years under a policy issued currently *on the assumption that the current dividend scale will not be changed.*

S' is the cash surrender value available at the end of n years under a policy issued currently.

n is the number of years covered by the net cost study and is usually either 10 or 20.

In history net cost, the figures reflect the actual results produced by the company in question. In net cost based on the current dividend scale, the figures do not reflect any actual results and will be borne out by actual results only in the unlikely event that mortality, interest, and expenses remain sufficiently constant so that the company will not change its dividend scale before the n years have elapsed.

It should be noted that the popular definitions of net cost ignore interest and mortality. Two dividend scales may be actuarially equivalent (i.e., the present value of one set of dividends discounted at interest and mortality may be exactly equal to the present value of a second set) but not have the same sum.

${}_nk_x$ is the symbol to denote the net single premium payable at the *end* of the term for n-year term insurance of \$1. ${}_nk_x$ is the accumulated cost of insurance in the technical sense. Issuing a policy on this basis would be mathematically sound but would have no practical value to a commercial life insurance company because of the difficulty of collecting the premium from the survivors after their insurance has expired. The concept of accumulated cost is, however, very valuable in connection with reserve calculations, as will be seen in the next chapter.

Since the only difference between ${}_nk_x$ and $A^1_{x:\overline{n}|}$ is that ${}_nk_x$ is the value of the term benefit at age $x + n$ and $A^1_{x:\overline{n}|}$ is the value of the same benefit at age x, we have the relation

$${}_nk_x \cdot {}_nE_x = A^1_{x:\overline{n}|}$$

or, solving for ${}_nk_x$ and expressing the solution in commutation symbols,

$${}_nk_x = \frac{M_x - M_{x+n}}{D_{x+n}}. \tag{42}$$

If $n = 1$, it is usually not written. Thus,

$$k_x = \frac{M_x - M_{x+1}}{D_{x+1}} = \frac{C_x}{D_{x+1}}. \tag{43}$$

Numerical values of $1000{}_nk_x$ and $1000k_x$ are found in Tables 12 and 5, respectively. Actually $1000k_x$ is in both tables. The excuse for the existence of Table 5 is that in Fackler's method for calculating reserves

u_x and k_x are used together; it is convenient to have them side by side in a table. Fackler's method is described in the next chapter.

28 • GENERAL INSURANCE FORMULA

We can now write down a general insurance formula similar to the general annuity formula of section 19.

$$R \cdot \frac{M_x - M_{x+n}}{D_y} \tag{44}$$

represents the value at age y of an insurance benefit of R payable at the end of the year in which an individual dies, provided he dies between ages x and $x + n$.

This formula does not apply if an endowment benefit is involved.

PROBLEM SET 18

1. Find the net annual premium for a \$1000 whole-life policy issued at age 20 providing for 10 annual premiums if the insurance coverage does not become effective until age 30.
2. Derive formulas 41 by the discount method.
3. Prove the identities:

$$(a) \quad {}_{r|}A_x = A_x - A^{1}_{x:\overline{r|}}.$$

$$(b) \quad {}_{r|}A^{1}_{x:\overline{n|}} = A^{1}_{x:\overline{n+r|}} - A^{1}_{x:\overline{r|}}.$$

$$(c) \quad {}_{r|}A_{x:\overline{n|}} = A_{x:\overline{n+r|}} - A^{1}_{x:\overline{r|}}.$$

4. Interpret the identities in problem 3 verbally.
5. A life insurance policy issued at age 32 provides for 28 annual premiums. If the insured dies before age 65, the death benefit is \$5000; if he dies after age 65, the death benefit is \$2000. Calculate the net annual premium.
6. A life insurance policy issued at age 22 provides for 10 annual premiums. The death benefit is \$1000 for the first 15 years and \$2000 thereafter. Find the net annual premiums.
7. Various forms of *modified life* policies are issued by insurance companies. A common form provides that, for a whole-life benefit, the premium for the first 5 years is half of the ultimate premium (i.e., the premium for the sixth and subsequent years). Find the ultimate net premium for such a policy issued to a person aged 45 with a face amount of \$1000.
8. Prove the following identities:

$$(a) \quad A^{1}_{x:\overline{n|}}\left(1 + \frac{1}{{}_nk_x}\right) = A_{x:\overline{n|}}.$$

$$(b) \quad A_x \cdot u_x = k_x + A_{x+1}.$$

$$(c) \quad A_{x:\overline{n|}}u_x = k_x + A_{x+1:\overline{n-1|}}.$$

$$(d) \quad c_xu_x = k_x.$$

(e) $P^{1}_{x:\overline{n}|} \cdot {}_nu_x = {}_nk_x.$

(f) ${}_nk_x = \dfrac{{}_nu_x}{1+i} - \dfrac{a_{x:\overline{n}|}}{{}_nE_x}.$

(g) $P_x \cdot u_x - k_x = A_{x+1} - P_x\ddot{a}_{x+1}.$

9. Give verbal interpretations for problems 8*b* and 8*e*.

PROBLEM SET 19

1. Find the net single premium for a life insurance policy providing a \$2000 death benefit for 27 years and \$1000 thereafter if the age at issue is 33.

2. If premiums are limited to 20 years, find the net annual premium for the policy described in question 1.

3. Find the net single premium for a life insurance policy providing a \$1000 death benefit for 10 years followed by \$2000 for 10 more years if the age at issue is 40.

4. Find the net single premium for a life insurance policy providing a \$2000 death benefit for 10 years followed by \$1000 for 10 more years if the age at issue is 40.

5. Explain in general terms why the answer to problem 3 is greater than the answer to problem 4.

6. Calculate the net annual premium for the policy in problem 4 if premiums are payable to age 65.

7. Discuss the policy of problem 6 from the standpoint of practicality.

8. Calculate the net single premium for 3-year term insurance of \$1000 issued at age 19.

9. Work problem 8 on the assumption of 2% interest instead of 2½%.

10. Prove that $A_x + a_x < \ddot{a}_x$.

C H A P T E R   F I V E

Net Level Reserves

29 · INTRODUCTION

A person insured under a 1-year renewable term policy would be paying the exact cost of his insurance each year. Since in the CSO table, and all other mortality tables, the probability of dying increases with age except at the very young ages, such a person would find that his premium increased every year. Net level premiums are designed to eliminate these increases. Since the death benefits are the same under a 1-year renewable term policy with no limit on the number of renewals as under an ordinary life policy, the present value (at issue) of the term premiums must equal the value at issue of the ordinary life premiums. Since the term premiums are increasing, they must be less at the start, and eventually become greater, than the ordinary life premiums. That this is actually the case can be seen by comparing the $1000c_x$ column in Table 5 with the $1000P_x$ column in Table 13. Since the ordinary life premium is more than enough to pay the cost of insurance for the face amount in the early years, a fund is created from the excess payments together with compound interest thereon. This fund is called the reserve. When a policy becomes a death claim, the reserve for the policy makes up part of the total payment. The difference between the face amount and the reserve is the actual "insurance" and is called the *net amount at risk*.

The way the reserve works can be seen from the following illustration. Consider a \$1000 ordinary life policy issued at age 95. (The high age is chosen to save arithmetic.) The net annual premium is $1000M_{95}/N_{95} = \$455.37$. Suppose that such a policy is issued to $l_{95}(= 3011)$ persons. Then the total premium would be $3011 \times 455.37 = \$1,371,119.07$. With interest at $2\frac{1}{2}\%$ this would amount to \$1,405,397.05 at the end of the year. During the year, d_{95} (= 1193) persons would die. Paying the claims at the end of the year would take \$1,193,000, leaving \$212,397.05 in the fund. If this amount were divided among the l_{96} (= 1818) survivors, each survivor's share would

be $116.83, which is the first-year *terminal reserve* for a $1000 ordinary life policy issued at age 95. The following table carries the illustration through until all policyholders have died.

(1) Year	1	2	3	4	5
(2) Age at start of year	95	96	97	98	99
(3) Number living at start of year	3011	1818	1005	454	125
(4) Number dying during year	1193	813	551	329	125
(5) Net premiums paid	$1371,119.07	$ 827,862.66	$457,646.85	$206,737.98	$ 56,921.25
(6) Fund at start of year = (5) + last year's (9)	1371,119.07	1040,259.71	710,913.05	384,423.86	121,955.71
(7) Fund at end of year before payment of death claims = (6) × 1.025	1405,397.05	1066,266.20	728,685.88	394,034.46	125,004.60
(8) Death claims	1193,000.00	813,000.00	551,000.00	329,000.00	125,000.00
(9) Total reserve at end of year	212,397.05	253,266.20	177,685.88	65,034.46	4.60*
(10) Policy terminal reserve	116.83	252.01	391.38	520.28	

* Theoretically, this is zero. The reason it isn't here is that all our calculations were chopped off to the nearest cent.

The above illustration attacks the problem by looking backwards, in time, to see what has happened. This approach is an example of the *retrospective* method.

The reserve may also be explained by looking forward in time. Net premiums are calculated so as to satisfy the fundamental relation that the present value of the future premiums is equal to the present value of the future benefits. But this relation holds only at the date of issue. At any time after the date of issue the value of the remaining net premiums will be less than the value of all the net premiums at the date of issue, because fewer premiums remain to be paid; also the value of the benefits will have increased because the time of payment has drawn nearer. Therefore, at any time after the date of issue the value of future benefits exceeds the value of future premiums. The insurance company must have enough money to make up the difference. This amount is called the reserve. This method of considering the reserve by looking forward in time is called the *prospective* method.

30 · RETROSPECTIVE RESERVE

In the last section a calculation of the retrospective reserve was illustrated. This method is valuable from the pedagogical standpoint but is too cumbersome to be worth much in practice. A more efficient method will now be considered.

From the retrospective point of view the terminal reserve is the accumulated value of past net premiums minus the accumulated value

of past insurance benefits. This can be expressed algebraically as follows:

$$_tV = P \cdot {}_tu_x - {}_tk_x \tag{45}$$

where x is the age at issue.

t is the number of years elapsed since issue.

P is the net level annual premium for a policy issued at age x with a face amount of \$1.

$_tV$ is the terminal reserve for the policy at the end of t years.

The reader will observe that in equation 45 both the net premiums and the cost of insurance have been accumulated with the benefit of interest and survivorship. At first glance this may not seem consistent with the method used in the illustration of section 29. We shall, therefore, demonstrate that the two methods are identical.

According to the method of section 29,

$$_1V = \frac{l_xP(1+i) - d_x}{l_{x+1}},$$

$$_2V = \frac{(l_{x+1} \cdot {}_1V + l_{x+1} \cdot P)(1+i) - d_{x+1}}{l_{x+2}},$$

$$_3V = \frac{(l_{x+2} \cdot {}_2V + l_{x+2} \cdot P)(1+i) - d_{x+2}}{l_{x+3}},$$

and, in general,

$$_tV = \frac{(l_{x+t-1} \cdot {}_{t-1}V + l_{x+t-1} \cdot P)(1+i) - d_{x+t-1}}{l_{x+t}}.$$

The general expression can be simplified.

$$\begin{aligned} _tV &= \frac{l_{x+t-1}}{l_{x+t}}({}_{t-1}V + P)(1+i) - \frac{d_{x+t-1}}{l_{x+t}} \\ &= \frac{v^{x+t-1}l_{x+t-1}}{v^{x+t}l_{x+t}}({}_{t-1}V + P) - \frac{v^{x+t}d_{x+t-1}}{v^{x+t}l_{x+t}} \\ &= \frac{D_{x+t-1}}{D_{x+t}}({}_{t-1}V + P) - \frac{C_{x+t-1}}{D_{x+t}} \end{aligned}$$

or

$$_tV = ({}_{t-1}V + P)u_{x+t-1} - k_{x+t-1} \tag{46}$$

or, since t is perfectly general,

$$_{t+1}V = ({}_tV + P)u_{x+t} - k_{x+t}. \tag{47}$$

Letting $t = 1, 2, 3, \cdots$ in equation 46, we have

$$
\begin{aligned}
{}_1V &= Pu_x - k_x. \\
{}_2V &= ({}_1V + P)u_{x+1} - k_{x+1} \\
&= (Pu_x - k_x + P)u_{x+1} - k_{x+1} \\
&= Pu_xu_{x+1} - k_xu_{x+1} + Pu_{x+1} - k_{x+1} \\
&= P\frac{D_x}{D_{x+1}} \cdot \frac{D_{x+1}}{D_{x+2}} + P\frac{D_{x+1}}{D_{x+2}} - \frac{C_x}{D_{x+1}} \cdot \frac{D_{x+1}}{D_{x+2}} - \frac{C_{x+1}}{D_{x+2}} \\
&= P\frac{D_x + D_{x+1}}{D_{x+2}} - \frac{C_x + C_{x+1}}{D_{x+2}} \\
&= P\frac{N_x - N_{x+2}}{D_{x+2}} - \frac{M_x - M_{x+2}}{D_{x+2}} \\
&= P \cdot {}_2u_x - {}_2k_x. \\
{}_3V &= ({}_2V + P)u_{x+2} - k_{x+2} \\
&= (P \cdot {}_2u_x - {}_2k_x + P)u_{x+2} - k_{x+2} \\
&= P \cdot {}_3u_x - {}_3k_x
\end{aligned}
$$

and so on; in general

$$ {}_tV = P \cdot {}_tu_x - {}_tk_x $$

which is equation 45.

Note: The reader who is familiar with algebra can develop a mathematically rigorous proof for the proposition in question by using the principles of mathematical induction.

Example

Find by the retrospective method the tenth terminal reserve for a \$1000 20-pay, 30-year endowment issued at age 25.

Solution:

$$
\begin{aligned}
P\ddot{a}_{25:\overline{20}|} &= 1000A_{25:\overline{30}|} \\
P &= 1000\frac{M_{25} - M_{55} + D_{55}}{N_{25} - N_{45}} \\
&= \$32.81. \\
{}_{10}V &= P \cdot {}_{10}u_{25} - 1000 \cdot {}_{10}k_{25} \\
&= (32.81)(11.733579) - 39.99268 \\
&= \$344.99.
\end{aligned}
$$

Note: In actual practice P should he calculated to at least 4 decimals. Since many readers of this book will not have ready access to a calculating machine, P is calculated to the nearest cent in all examples in this chapter. Answers to the problems are calculated on that basis. The only exception is in connection with Fackler's method (section 33) where P is carried to 4 decimals.

PROBLEM SET 20

1. Find by the retrospective method the fifteenth terminal reserve for a \$1000 policy issued at age 20 under each of the following plans:

(*a*) Ordinary life.
(*b*) 20-pay life.
(*c*) Endowment at 65.
(*d*) Term to 65.
(*e*) 20-pay endowment at 60.

2. Find by the retrospective method the twelfth terminal reserve for a \$1000 18-pay life policy issued at age 30.

3. Express in commutation symbols the retrospective reserve for the tenth year for a \$1000 policy issued at age 30 under each of the following plans. Simplify the final expression as much as possible.

(*a*) Ordinary life.
(*b*) 15-pay life.
(*c*) 20-year term.
(*d*) Endowment at 70.
(*e*) 20-pay endowment at 85.
(*f*) 10-year term.
(*g*) 10-year endowment.
(*h*) 10-pay life.

4. Explain in words the results obtained for problems 3*f*, *g*, and *h*.

31 · NOTATION

In section 24 the recommendation was made that in working out the net premium for a particular policy we let P be the unknown net premium and solve the fundamental equation for P. This procedure is in lieu of using (at the time of working the prohlem) the standard symhols with numhers in the corners. The reason for the recommendation is that the use of correct corner symbols in a particular problem serves no apparent purpose. This does not mean that the authors think that the standard symbols are useless and should not be learned. On the contrary, in many situations the standard symhols are very useful; moreover, they must he learned hy anyone who plans to read any actuarial literature. The authors' recommendation regarding the use of an unadorned P refers only to the solutions of individual problems.

A similar recommendation is made in connection with reserves. It is recommended that in a particular problem ${}_tV$ be used to denote the tth terminal reserve for the policy under consideration. The standard symbols shown below should, however, be learned.

$_tV_x$ is the tth terminal reserve for a \$1 ordinary life policy issued at age x.

$_{t:n}V_x$ is the tth terminal reserve for a \$1 n-pay life policy issued at age x.

$_tV_{x:\overline{n}|}$ is the tth terminal reserve for a \$1 n-year endowment policy issued at age x.

$_tV^1_{x:\overline{n}|}$ is the tth terminal reserve for a \$1 n-year term policy issued at age x.

32 · PROSPECTIVE RESERVE

From the prospective point of view the insurance company must have enough money on hand so that this amount (the reserve) plus the present value of the future premiums at the valuation date equals the present value of future benefits at the valuation date. That is to say

$$_tV + (\text{Value at age } x + t \text{ of future net premiums}) = (\text{Value at age } x + t \text{ of future benefits}),$$

or

$$_tV = (\text{Value at age } x + t \text{ of future benefits}) - (\text{Value at age } x + t \text{ of future net premiums}), \quad (48)$$

where x is the age at issue.

Example 1

Find by the prospective method the tenth terminal reserve for a \$1000 20-pay, 30-year endowment issued at age 25.

Solution: From the example in section 30, $P = \$32.81$.

$$_{10}V = 1000A_{35:\overline{20}|} - P\ddot{a}_{35:\overline{10}|}$$

$$= 632.50285 - (32.81)(8.765669)$$

$$= \$344.90 \text{ compared with } \$344.99 \text{ by the retrospective method.}$$

Note: The 9-cent discrepancy is caused by taking P to the nearest cent.

Example 2

Find by both the retrospective and prospective methods the fifteenth terminal reserve for a \$1000 10-pay life policy issued at age 20.

Solution: (*a*) By the retrospective method.

$$P\ddot{a}_{20:\overline{10|}} = 1000A_{20},$$

$$P = \frac{1000A_{20}}{\ddot{a}_{20:\overline{10|}}}$$

$$= \frac{338.67727}{8.869069}$$

$$= \$38.19.$$

It is tacitly assumed in formula 45 that premiums are payable for at least t years. This is not the case in the present problem. We must make a modification in the method.

$$_{15}V = P\,\frac{N_{20} - N_{30}}{D_{35}} - 1000 \cdot {}_{15}k_{20}$$

$$= (38.19)(13.48166) - 58.20308$$

$$= \$456.66.$$

(*b*) By the prospective method. Since the policy is paid up at the date of reserve calculation, the present value of future premiums is nil; hence, the reserve is the present value of future benefits.

$$_{15}V = 1000A_{35}$$

$$= \$456.61.$$

Note: It is almost always easier to calculate the reserve for a paid-up policy by the prospective method, which should, therefore, be used.

In example 2 the fifteenth terminal reserve for the 10-pay life policy was the same for both methods except for a minor difference caused by dropping places. Example 1 and the example of section 30 also gave identical results for the two methods when applied to a 20-pay, 30-year endowment. The identical results found in these examples are not, of course, coincidence. Barring slight differences due to rounding, the prospective reserve is always equal to the retrospective reserve.* It is frequently desirable to work out a problem by both methods, thus giving an independent check. In actual office practice, of course, enough places are carried so that the results will be the same to the nearest cent.

* The equivalence of the prospective and retrospective reserves can be proved by the following line of reasoning. In calculating the net premium the funda-

PROBLEM SET 21

1. Find by the prospective method the tenth terminal reserve for a $1000 policy issued at age 28 under each of the following plans:

(a) Ordinary life.
(b) 20-pay life.
(c) 30-year endowment.
(d) 20-pay endowment at 65.
(e) Single-premium endowment at 60.

2. Express in commutation symbols the prospective reserve for the twelfth year for a $1000 policy issued at age 40 under each of the following plans. Simplify the final expression as much as possible.

(a) Ordinary life.
(b) 15-pay life.
(c) 20-year endowment.
(d) 10-pay life.
(e) 15-year term.
(f) 10-pay endowment at 70.

3. Prove that ${}_tV_x$ can be written in each of the following forms:

$$(a)\quad 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}.$$

$$(b)\quad \frac{A_{x+t} - A_x}{1 - A_x}.$$

$$(c)\quad A_{x+t}\left(1 - \frac{P_x}{P_{x+t}}\right).$$

---

mental equation is

(Value at date of issue of all net premiums) = (Value at date of issue of all policy benefits).

Accumulating both sides of this equation with the benefit of interest and survivorship to the date at which the reserve is to be calculated, we have

(Value at reserve date of all net premiums) = (Value at reserve date of all policy benefits).

The value of all net premiums at the reserve date may be separated into two parts: (a) the accumulated value of all net premiums paid before the reserve date and (b) the discounted value of all future net premiums. Similarly, the value of all policy benefits at the reserve date may be divided into two parts: (a) the accumulated value of the temporary insurance provided by the policy prior to the reserve date and (b) the discounted value of all future policy benefits. Hence, as of the reserve date, we have

(Accumulated value of past premiums) + (Discounted value of future premiums) = (Accumulated value of past benefits) + (Discounted value of future benefits).

Or

(Accumulated value of past premiums) − (Accumulated value of past benefits) = (Discounted value of future benefits) − (Discounted value of future premiums).

Therefore, from equations 45 and 48,

(Retrospective reserve) = (Prospective reserve).

4. Prove the following identities:

$$(a)\quad {}_tV_{x:\overline{n|}} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t|}}}{\ddot{a}_{x:\overline{n|}}}.$$

$$(b)\quad {}_{t:n}V_x = ({}_{n-t}P_{x+t} - {}_nP_x)\ddot{a}_{x+t:\overline{n-t|}}.$$

$$(c)\quad {}_tV^1_{x:\overline{n|}} = (P^{\ 1}_{x+t:\overline{n-t|}} - P^1_{x:\overline{n|}})\ddot{a}_{x+t:\overline{n-t|}}.$$

$$(d)\quad {}_tV_{x:\overline{n|}} = (P_{x+t:\overline{n-t|}} - P_{x:\overline{n|}})\ddot{a}_{x+t:\overline{n-t|}}.$$

5. Give a verbal interpretation of problems 4*b*, *c*, and *d*.

6. Prove that

$${}_tV_x = 1 - (1 - {}_1V_x)(1 - {}_1V_{x+1})(1 - {}_1V_{x+2}) \cdots (1 - {}_1V_{x+t-1}).$$

7. If $\ddot{a}_x + \ddot{a}_{x+2t} = 2\ddot{a}_{x+t}$ and ${}_tV_x = \frac{1}{4}$, find the numerical values of ${}_tV_{x+t}$ and ${}_{2t}V_x$.

33 • FACKLER'S METHOD

It was demonstrated in section 30 that the retrospective reserve as represented by formula 45 is identical with the method used in section 29. In making this demonstration, equation 47,

$${}_{t+1}V = ({}_tV + P)u_{x+t} - k_{x+t},$$

was derived incidentally. This equation is referred to as Fackler's accumulation formula. Calculating reserves by means of this formula is called Fackler's method. Fackler's method is a very valuable tool for calculating the reserves for a particular policy with a given age at issue for several consecutive durations. The example below will illustrate the method.

Example

Calculate by Fackler's method the first five terminal reserves for a $1000 20-pay, 30-year endowment issued at age 30. Check by the prospective method.

Solution: Letting t take on all values from 0 to 4 inclusive in formula 47, we have

$$\begin{aligned}
{}_1V &= Pu_{30} - 1000k_{30},\\
{}_2V &= ({}_1V + P)u_{31} - 1000k_{31},\\
{}_3V &= ({}_2V + P)u_{32} - 1000k_{32},\\
{}_4V &= ({}_3V + P)u_{33} - 1000k_{33},\\
{}_5V &= ({}_4V + P)u_{34} - 1000k_{34}.
\end{aligned}$$

All we have to do is to calculate P, substitute the calculated value in the expression for ${}_1V$ and continue from there.

$$P = \frac{1000A_{30:\overline{30}|}}{\ddot{a}_{30:\overline{20}|}}$$

$$= \$33.7900.$$

If we are working the problem without a calculating machine, we can consider the two zeros on the end as a lucky break.

$$
\begin{aligned}
{}_1V &= (33.7900)(1.0286625) - 3.57315 \\
&= 31.1854; \\
{}_2V &= (64.9754)(1.0288381) - 3.74450 \\
&= 63.1047; \\
{}_3V &= (96.8947)(1.0290337) - 3.93533 \\
&= 95.7726; \\
{}_4V &= (129.5626)(1.0292407) - 4.13722 \\
&= 129.2139; \\
{}_5V &= (163.0039)(1.0294785) - 4.36929 \\
&= 163.4397.
\end{aligned}
$$

Check:

$$
\begin{aligned}
{}_5V &= 1000A_{35:\overline{25}|} - P\ddot{a}_{35:\overline{15}|} \\
&= 575.73071 - 412.29186 \\
&= 163.4389,
\end{aligned}
$$

which agrees with ${}_5V$ as calculated by Fackler's method except for a difference of 0.0008, which is caused by the accumulation of very small rounding errors. Since every reserve calculated by Fackler's method depends on the reserves ahead of it, we can conclude that, since ${}_5V$ checks, the other reserves must be right—barring compensating errors.

PROBLEM SET 22

1. Calculate and check the first five terminal reserves for a \$10,000 15-pay life policy issued at age 37.

2. Calculate by Fackler's method the reserves for a 4-pay, 6-year endowment policy for \$1000 issued at age 25.

3. A \$1000 endowment at 65 issued at age 20 provides that the premium for the first 3 years shall be 80% of the premium for the fourth and subsequent years.

Use Fackler's method to calculate the first five terminal reserves. Check by the prospective method.

4. Give a verbal proof of the equation

$${}_{r+t}V = {}_tV \cdot \frac{1}{{}_rE_{x+t}} + P \cdot {}_ru_{x+t} - {}_rk_{x+t}.$$

5. Using the results of problem 1 and the equation of problem 4, compute the tenth terminal reserve for a $1000 15-pay life policy issued at age 37.

34 • INITIAL RESERVE, MEAN RESERVE

The terminal reserve for a given policy for a given policy year is defined as the reserve at the end of that year, just before the premium for the next year has been paid. The *initial reserve* is defined as the reserve at the start of the year just after the premium for that year has been paid. From these definitions we have the relation

$${}_tI = {}_{t-1}V + P,$$

where ${}_tI$ is the initial reserve for the tth year. In case no premium is due at the start of the tth year the initial reserve is equal to the last year's terminal reserve; that is

$${}_tI = {}_{t-1}V.$$

The *mean reserve* is defined as the arithmetic mean of the initial and terminal reserves. ${}_t(MV)$ is ordinarily used to denote the mean reserve for the tth policy year. Thus

$${}_t(MV) = \frac{{}_tI + {}_tV}{2} = \frac{{}_{t-1}V + P + {}_tV}{2},$$

where P represents the net premium due at the start of the tth policy year and may, of course, be zero.

Life insurance companies are required to file an annual statement with the various state insurance departments. This statement shows, among other things, the total reserves as of December 31 on all outstanding policies. For convenience in performing the calculations it is assumed that all policies issued in a given calendar year were issued on July 1 of that year. Since December 31 is in the middle of the policy year beginning July 1, the mean reserve is used as the December 31 (annual statement) reserve. For the December 31, 1950, statement the first mean reserve would be used for all policies issued in 1950, the second mean reserve for all policies issued in 1949, and so on.

An interesting and instructive relationship connecting the initial

and terminal reserves and the net amount at risk can be derived from Fackler's accumulation formula.

$$
\begin{aligned}
{}_{t+1}V &= ({}_tV + P)u_{x+t} - k_{x+t} \\
&= {}_{t+1}I\,\frac{D_{x+t}}{D_{x+t+1}} - \frac{C_{x+t}}{D_{x+t+1}} \\
&= {}_{t+1}I\,\frac{v^{x+t}l_{x+t}}{v^{x+t+1}l_{x+t+1}} - \frac{v^{x+t+1}d_{x+t}}{v^{x+t+1}l_{x+t+1}} \\
&= \frac{{}_{t+1}I(1+i)}{p_{x+t}} - \frac{l_{x+t} - l_{x+t+1}}{l_{x+t+1}} \\
&= \frac{{}_{t+1}I(1+i)}{p_{x+t}} - \frac{1}{p_{x+t}} + 1.
\end{aligned}
$$

Dropping the corner symbols from I and V and multiplying both sides of the equation by p_{x+t}, we have

$$p_{x+t} \cdot V = I(1+i) - 1 + p_{x+t}$$

or

$$(1 - q_{x+t})V = I(1+i) - q_{x+t},$$

whence

$$V = I(1+i) - q_{x+t}(1 - V). \tag{49}$$

The term $q_{x+t}(1 - V)$ is the product of the probability of dying in the $(t+1)$st policy year, and the net amount at risk in the $(t+1)$st policy year, and is called the cost of insurance based upon the net amount at risk. Equation 49 says, in words, that the terminal reserve for any year is the initial reserve for that year accumulated at interest and decreased by the cost of insurance based upon the net amount at risk.

PROBLEM SET 23

1. Calculate the December 31, 1951, reserve for \$1000 policies issued in 1941 at age 30 under the following plans:

(*a*) Ordinary life.
(*b*) 20-year endowment.
(*c*) 10-pay life.
(*d*) Single-premium endowment at 65.

2. For a \$1000 policy issued at age 40 the cost of insurance based on the net amount at risk for the tenth policy year is \$5 Find the tenth terminal reserve.

3. Find the tenth mean reserve for an annual-premium deferred life annuity issued at age 27 where the annual annuity payments are \$100, the first payment is due at age 65, and no refund is made at the death of the insured.

4. Find the first mean reserve and the cost of insurance based on the net amount at risk for a 20-pay life policy for \$2000 issued at age 18.

PROBLEM SET 24

1. Calculate the tenth terminal reserve for a \$5000 ordinary life policy issued at age 21.

2. Calculate the first five terminal reserves for a \$1000 20-pay life policy issued at age 22.

3. Calculate the first five initial reserves for a \$1000 20-pay life policy issued at age 22.

4. Calculate the first five mean reserves for a \$1000 20-pay life policy issued at age 22.

5. Calculate the first five mean reserves for a \$10,000 single-premium 30-year endowmeot policy issued at age 35.

6. A policy provides for a death benefit of \$3000 during the first 20 years and \$1000 thereafter. Premiums are payable for life. If the age at issue is 25, calculate the third, twentieth, and thirtieth terminal reserves.

7. Find the twentieth mean reserve under a \$10,000 life paid-up at 60 policy issued at age 45.

8. Express ${}_{8:10}V_{20}$ in commutation symbols.

9. Express ${}_{15:10}V_{20}$ in commutation symbols.

10. Given: $5u_x = 51k_x$ and $i = .02$. Find p_x.

C H A P T E R S I X

Advanced Topics

35 • ANNUITIES CERTAIN PAYABLE FRACTIONALLY

$a_{\overline{n}|}^{(m)}$ represents the present value of an annuity certain under which payments are made m times a year for n years with each payment equal to $1/m$ and the first one being due $1/m$th of a year from now. Such an annuity is usually described as an annuity certain immediate payable m times a year to run for n years with annual rent \$1. The term "annual rent" is a technical term used in connection with annuities and refers to the total payment during 1 year. Similarly $\ddot{a}_{\overline{n}|}^{(m)}$ represents the present value of an annuity certain due payable m times a year to run for n years with annual rent \$1.

Since $a_{\overline{n}|}^{(m)}$ is the sum of the present values of the mn individual payments, we have

$$\begin{aligned} a_{\overline{n}|}^{(m)} &= \frac{1}{m}[(1+i)^{-\frac{1}{m}} + (1+i)^{-\frac{2}{m}} + (1+i)^{-\frac{3}{m}} + \cdots \\ &\qquad + (1+i)^{-1} + (1+i)^{-1-\frac{1}{m}} + \cdots + (1+i)^{-n}] \\ &= \frac{1}{m}[(1+i)^{\frac{m-1}{m}} + (1+i)^{\frac{m-2}{m}} + (1+i)^{\frac{m-3}{m}} + \cdots \\ &\qquad + (1+i)^{\frac{1}{m}} + 1] \times [(1+i)^{-1} + (1+i)^{-2} + (1+i)^{-3} + \\ &\qquad \cdots + (1+i)^{-n}]. \end{aligned}$$

Denoting $\frac{1}{m}[(1+i)^{\frac{m-1}{m}} + (1+i)^{\frac{m-2}{m}} + (1+i)^{\frac{m-3}{m}} + \cdots + 1]$ by $s_{\overline{1}|}^{(m)}$ and writing v for $(1+i)^{-1}$, we have

$$a_{\overline{n}|}^{(m)} = (v + v^2 + v^3 + \cdots + v^n)s_{\overline{1}|}^{(m)}$$

or

$$a_{\overline{n}|}^{(m)} = a_{\overline{n}|}s_{\overline{1}|}^{(m)} \quad \text{from equation 4.} \tag{50}$$

Since the only difference between $a_{\overline{n}|}^{(m)}$ and $\ddot{a}_{\overline{n}|}^{(m)}$ is that each payment under the latter is made $1/m$ of a year earlier than the corresponding

payment under the former, we have

$$\ddot{a}_{\overline{n}|}^{(m)} = (1+i)^{\frac{1}{m}} a_{\overline{n}|}^{(m)} = a_{\overline{n}|} s_{\overline{1}|}^{(m)} (1+i)^{\frac{1}{m}}. \tag{51}$$

The table below gives values of $s_{\overline{1}|}^{(m)}$, $(1+i)^{\frac{1}{m}}$ and $s_{\overline{1}|}^{(m)}(1+i)^{\frac{1}{m}}$ for $m = 2, 4,$ and 12 and $i = 2\frac{1}{2}\%$.

m	$s_{\overline{1}|}^{(m)}$	$(1.025)^{\frac{1}{m}}$	$s_{\overline{1}|}^{(m)}(1.025)^{\frac{1}{m}}$
2	1.006 2114	1.012 4228	1.018 7114
4	1.009 3268	1.006 1922	1.015 5768
12	1.011 4072	1.002 0598	1.013 4905

PROBLEM SET 25

1. Find the present value of an annuity certain immediate for $100 every quarter to run for 20 years.

2. Find the present value of an annuity certain immediate with annual rent $300 payable monthly to run for 15 years.

3. Find the present value of an annuity certain due for $10 per month to run for 10 years.

4. Calculate the net single premium for a whole-life insurance policy issued at age 30, providing that at the death of the insured 240 monthly payments of $100 each will be made by the insurance company, the first payment being due at the end of the policy year in which the insured dies.

5. Give a verbal proof of the equation

$$\ddot{s}_{\overline{n}|}^{(m)} = \ddot{a}_{\overline{n}|}^{(m)}(1+i)^n.$$

36 • LIFE ANNUITIES PAYABLE FRACTIONALLY

As in annuities certain the superscript (m) means that the annuity is payable m times a year. Thus, $\ddot{a}_x^{(m)}$ denotes the present value of a whole-life annuity due to (x) payable m times a year with an annual rent of $1. Going back to first principles, we have

$$\ddot{a}_x^{(m)} = \frac{1}{m}\left(1 + v^{\frac{1}{m}} \cdot {}_{\frac{1}{m}}p_x + v^{\frac{2}{m}} \cdot {}_{\frac{2}{m}}p_x + \cdots\right).$$

Producing a numerical answer for this expression involves a fantastic amount of arithmetic. As a practical matter it is customary to use an approximation; namely, $\ddot{a}_x - \dfrac{m-1}{2m}$. The justification for this approximation is discussed below.

Consider the two identities:

$${}_{0|}\ddot{a}_x = \ddot{a}_x - 0, \quad {}_{1|}\ddot{a}_x = \ddot{a}_x - 1.$$

Linear interpolation between these values gives the relations

$$_{\frac{1}{m}|}\ddot{a}_x = \ddot{a}_x - \frac{1}{m},$$

$$_{\frac{2}{m}|}\ddot{a}_x = \ddot{a}_x - \frac{2}{m},$$

and, in general,

$$_{\frac{k}{m}|}\ddot{a}_x = \ddot{a}_x - \frac{k}{m}.$$

Now $_{\frac{1}{m}|}\ddot{a}_x$ represents a life annuity to (x) with annual payments of \$1, the first payment due in $1/m$th of a year, the second due in $\left(1 + \frac{1}{m}\right)$ years, etc. Similarly, $_{\frac{2}{m}|}\ddot{a}_x$ represents a life annuity to (x) with annual payments of \$1, the first payment due $2/m$th of a year from now, the second due in $\left(1 + \frac{2}{m}\right)$ years, etc. Apparently

$$_{0|}\ddot{a}_x + {_{\frac{1}{m}|}\ddot{a}_x} + {_{\frac{2}{m}|}\ddot{a}_x} + \cdots + {_{\frac{m-1}{m}|}\ddot{a}_x}$$

represents the present value of a series of payments of \$1 payable at intervals of $1/m$th of a year, the first payment to be made right now and payments to continue as long as (x) is alive. But this series of payments is a life annuity due payable m times a year with annual rent m dollars. Hence,

$$\begin{aligned}
m \cdot \ddot{a}_x^{(m)} &= {_{0|}\ddot{a}_x} + {_{\frac{1}{m}|}\ddot{a}_x} + {_{\frac{2}{m}|}\ddot{a}_x} + \cdots + {_{\frac{m-1}{m}|}\ddot{a}_x} \\
&= (\ddot{a}_x - 0) + \left(\ddot{a}_x - \frac{1}{m}\right) + \left(\ddot{a}_x - \frac{2}{m}\right) + \cdots \\
&\qquad + \left(\ddot{a}_x - \frac{m-1}{m}\right) \\
&= m\ddot{a}_x - \frac{1}{m}[1 + 2 + 3 + \cdots + (m-1)] \\
&= m\ddot{a}_x - \frac{1}{m} \cdot \frac{(m-1)m}{2} \\
&= m\ddot{a}_x - \frac{m-1}{2}.
\end{aligned}$$

Hence,

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m}. \qquad (52)$$

It should be remembered that equation 52 expresses an approximate relation so that theoretically the word "approximately" should be tacked on at the end of this and similar equations. The authors will dispense with this formality.

Since the only difference between the payments represented by $\ddot{a}_x^{(m)}$ and $a_x^{(m)}$ is the payment of $1/m$ due right now under $\ddot{a}_x^{(m)}$, we have

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} = \ddot{a}_x - \frac{m+1}{2m} = a_x + \frac{m-1}{2m}. \tag{53}$$

The most usual, and therefore the most important, fractional periods are semiannual ($m = 2$), quarterly ($m = 4$), and monthly ($m = 12$). When $m = 2$, 4, or 12, we have the following formulas:

$$\ddot{a}_x^{(2)} = \ddot{a}_x - \tfrac{1}{4}; \qquad a_x^{(2)} = a_x + \tfrac{1}{4};$$

$$\ddot{a}_x^{(4)} = \ddot{a}_x - \tfrac{3}{8}; \qquad a_x^{(4)} = a_x + \tfrac{3}{8};$$

$$\ddot{a}_x^{(12)} = \ddot{a}_x - \tfrac{11}{24}; \qquad a_x^{(12)} = a_x + \tfrac{11}{24}.$$

For a deferred life annuity due we have

$$_{n|}\ddot{a}_x^{(m)} = {}_nE_x\ddot{a}_{x+n}^{(m)}$$

or

$$_{n|}\ddot{a}_x^{(m)} = {}_nE_x\left(\ddot{a}_{x+n} - \frac{m-1}{2m}\right). \tag{54}$$

For a temporary life annuity due we have

$$\begin{aligned}\ddot{a}_{x:\overline{n|}}^{(m)} &= \ddot{a}_x^{(m)} - {}_{n|}\ddot{a}_x^{(m)} \\ &= \ddot{a}_x - \frac{m-1}{2m} - {}_nE_x\left(\ddot{a}_{x+n} - \frac{m-1}{2m}\right) \\ &= (\ddot{a}_x - {}_nE_x\ddot{a}_{x+n}) - \frac{m-1}{2m}(1 - {}_nE_x)\end{aligned}$$

or

$$\ddot{a}_{x:\overline{n|}}^{(m)} = \ddot{a}_{x:\overline{n|}} - \frac{m-1}{2m}(1 - {}_nE_x). \tag{55}$$

Example 1

Find the present value of a life annuity due of \$100 per month to a man aged 65 if the first 120 payments will be paid whether the man lives or dies.

Note: This type of annuity is frequently called "10 years certain and continuous" because the payments for the first 10 years are guaranteed and later payments will continue so long as the annuitant shall live.

Solution:

$$\begin{aligned}
\text{Present value} &= 1200\ddot{a}_{\overline{10}|}^{(12)} + 1200 \cdot {}_{10|}\ddot{a}_{65}^{(12)} \\
&= 1200a_{\overline{10}|}s_{\overline{1}|}^{(12)}(1.025)^{1/12} + 1200 \cdot {}_{10}E_{65}(\ddot{a}_{75} - 11/24) \\
&= (1200)(8.752064)(1.0134905) \\
&\qquad + (1200)\left(\frac{49587.53}{116088.15}\right)\left(6.546383 - \frac{11}{24}\right) \\
&= 10{,}644.160 + 3120.643 \\
&= \$13{,}764.80.
\end{aligned}$$

EXAMPLE 2

Calculate to the nearest cent the quarterly premium for a \$1000 term to age 65 policy issued at age 40.

Note: Two types of fractional premiums are used by United States life insurance companies. One type, which may be called the *installment fractional premium,* is calculated by solving the equation

$$mx \cdot \ddot{a}_{\overline{1}|}^{(m)} = P$$

where x is the fractional premium payable m times a year and P is the annual premium. Since probabilities of living and dying do not enter into the calculation, the death benefit is reduced by the value of the unpaid installments for the policy year in which death occurs.

The second type, which may be called the *true fractional premium,* is calculated by taking account of the probabilities of living and dying and does not involve a reduction of the death benefit because of unpaid fractional premiums in the year of death. The second type will be assumed in problems and examples in this book.

Solution: Let P be the quarterly premium. Then

$$4P\ddot{a}_{40:\overline{25}|}^{(4)} = 1000A_{40:\overline{25}|}^{1}.$$

But

$$\begin{aligned}
\ddot{a}_{40:\overline{25}|}^{(4)} &= \ddot{a}_{40:\overline{25}|} - 3/8(1 - {}_{25}E_{40}) \\
&= 16.828932 - 3/8(1 - 0.35286912) \\
&= 16.586258
\end{aligned}$$

and

$$1000A^{\;1}_{40:\overline{25|}} = 236.66912$$

so that

$$P = \frac{236.66912}{(4)(16.586258)}$$

$$= \$3.57$$

PROBLEM SET 26

1. Describe in words the annuities represented by the following symbols and evaluate them.

$$a^{(2)}_{30},\ \ddot{a}^{(12)}_{30},\ 1200\ddot{a}^{(12)}_{40:\overline{10|}},\ 40\ddot{a}^{(4)}_{35:\overline{20|}},\ {}_{10|}\ddot{a}^{(2)}_{50},\ 100a^{(4)}_{30:\overline{10|}},\ 13\ddot{a}^{(52)}_{30}.$$

2. Find the present value of a 5-year certain and continuous annuity due for $10 a month to a man aged 60.

3. Calculate the annual, semiannual, quarterly, and monthly net premiums for a 30-pay life policy for $1000 issued at age 24.

4. Calculate the quarterly premium for a $1000 modified life policy issued to (25) if premiums for the first 5 years are half of the premiums after 5 years.

5. A life annuity contract provides for a payment of $1200 every year, first payment at age 65. On attaining age 65, the annuitant decides that he would prefer to receive the payments monthly instead of annually. If the first monthly payment is to be made at age 65, calculate the equitable monthly payment.

6. Assuming a certain mortality table and rate of interest, the value of a life annuity of $100 payable at the end of every year to a certain individual is $1353.46.

(*a*) Find the present value of a life annuity of $10 payable at the end of every month to the same individual.

(*b*) Find the present value if the $10 payments are made at the beginning of every month.

7. Prove the following identities:

$$(a)\quad a^{(2)}_{x:\overline{n|}} = \tfrac{1}{4}(3a_{x:\overline{n|}} + \ddot{a}_{x:\overline{n|}}).$$

$$(b)\quad a^{(4)}_{x:\overline{n|}} = \tfrac{1}{8}(5a_{x:\overline{n|}} + 3\ddot{a}_{x:\overline{n|}}).$$

$$(c)\quad \ddot{a}^{(12)}_{x:\overline{1|}} = \frac{11vp_x + 13}{24}.$$

37 · INCREASING LIFE ANNUITIES

The symbol $(I\ddot{a})_x$ denotes the present value at age x of an increasing life annuity due under which the first payment is $1, the second payment is $2, the third payment is $3, and so on, the payments increasing by $1 every year. By going back to first principles and using the discount method, we have

$$
\begin{aligned}
(I\ddot{a})_x &= 1 + 2vp_x + 3v^2 \cdot {}_2p_x + 4v^3 \cdot {}_3p_x + \cdots \\
&= \frac{v^x l_x}{v^x l_x} + 2\,\frac{v^{x+1} l_{x+1}}{v^x l_x} + 3\,\frac{v^{x+2} l_{x+2}}{v^x l_x} + 4\,\frac{v^{x+3} l_{x+3}}{v^x l_x} + \cdots \\
&= \frac{D_x + 2D_{x+1} + 3D_{x+2} + 4D_{x+3} + \cdots + (\omega - x + 1)D_\omega}{D_x}.
\end{aligned}
$$

But since

$$
\begin{aligned}
N_x &= D_x + D_{x+1} + D_{x+2} + D_{x+3} + \cdots + D_\omega, \\
N_{x+1} &= \phantom{D_x + {}} D_{x+1} + D_{x+2} + D_{x+3} + \cdots + D_\omega, \\
N_{x+2} &= \phantom{D_x + D_{x+1} + {}} D_{x+2} + D_{x+3} + \cdots + D_\omega, \\
N_{x+3} &= \phantom{D_x + D_{x+1} + D_{x+2} + {}} D_{x+3} + \cdots + D_\omega, \\
&\cdots \\
N_\omega &= \phantom{D_x + D_{x+1} + D_{x+2} + D_{x+3} + \cdots + {}} D_\omega,
\end{aligned}
$$

we have, by adding the above equalities,

$$
\begin{aligned}
D_x + 2D_{x+1} + 3D_{x+2} + 4D_{x+3} + \cdots + (\omega - x + 1)D_\omega \\
= N_x + N_{x+1} + N_{x+2} + N_{x+3} + \cdots + N_\omega \\
= S_x
\end{aligned}
$$

by the definition in equations 19. Therefore

$$
(I\ddot{a})_x = \frac{S_x}{D_x}. \tag{56}
$$

Example 1

$(I\ddot{a})_{x:\overline{n}|}$ represents the present value at age x of a temporary increasing life annuity due under which the first payment is \$1, the second payment is \$2, the third payment is \$3, and so on, the last payment of \$$n$ being due at age $x + n - 1$. Prove that

$$
(I\ddot{a})_{x:\overline{n}|} = \frac{S_x - S_{x+n} - nN_{x+n}}{D_x}.
$$

Solution:

$$
(I\ddot{a})_{x:\overline{n}|} = \frac{D_x + 2D_{x+1} + 3D_{x+2} + \cdots + nD_{x+n-1}}{D_x},
$$

$$
\begin{aligned}
D_x(I\ddot{a})_{x:\overline{n}|} = {} & (D_x + D_{x+1} + D_{x+2} + \cdots + D_{x+n-1}) \\
& + (D_{x+1} + D_{x+2} + \cdots + D_{x+n-1}) + \cdots \\
& + (D_{x+n-2} + D_{x+n-1}) + (D_{x+n-1})
\end{aligned}
$$

$$= (N_x - N_{x+n}) + (N_{x+1} - N_{x+n}) + \cdots + (N_{x+n-2} - N_{x+n}) + (N_{x+n-1} - N_{x+n})$$

$$= (N_x + N_{x+1} + \cdots + N_{x+n-2} + N_{x+n-1}) - nN_{x+n}$$

$$= S_x - S_{x+n} - nN_{x+n},$$

or

$$(I\ddot{a})_{x:\overline{n}|} = \frac{S_x - S_{x+n} - nN_{x+n}}{D_x}.$$

Example 2

A man now aged 20 has a temporary life annuity calling for successive annual payments of $1000, $950, $900, $850, and so on, until the payments are reduced to $500 at which time they remain constant. The first payment is to be made immediately and the maximum number of payments is 20. Find the present value.

Solution: The successive annual payments may be broken down as shown in the following table:

Age	*Payment*
20	$1000
21	1000 − 50
22	1000 − 100
23	1000 − 150
.	
.	
.	
29	1000 − 450
30	1000 − 500
31	1000 − 500
.	
.	
.	
39	1000 − 500

From an inspection of this table we have the relation,

(Value of annuity in question) = (Value of a 20-year temporary life annuity for $1000 annually, first payment at age 20)—(Value of a 10-year temporary increasing life annuity, first payment of $50 due at age 21 and last payment of $500 due at age 30)—(Value of a temporary deferred life annuity for $500 annually, first payment at age 31 and last payment at age 39).

In symbols, letting x be the present value of the annuity in question, we have

$$x = 1000\frac{N_{20} - N_{40}}{D_{20}} - 50\frac{S_{21} - S_{31} - 10N_{31}}{D_{20}} - 500\frac{N_{31} - N_{40}}{D_{20}}$$

$$= \frac{1000N_{20} - 1000N_{40} - 50S_{21} + 50S_{31} + 500N_{31} - 500N_{31} + 500N_{40}}{D_{20}}$$

$$= \frac{1000N_{20} - 50S_{21} + 50S_{31} - 500N_{40}}{D_{20}}$$

$$= \$10{,}319.87.$$

Alternative solution: Using the principle that the present value of a series of payments is the sum of the present values of the individual payments, expressing these individual present values in terms of D's, and letting x be the present value of the series, we have

$$x = 1000\frac{D_{20}}{D_{20}} + 950\frac{D_{21}}{D_{20}} + 900\frac{D_{22}}{D_{20}} + \cdots + 550\frac{D_{29}}{D_{20}}$$
$$+ 500\frac{D_{30}}{D_{20}} + 500\frac{D_{31}}{D_{20}} + 500\frac{D_{32}}{D_{20}} + \cdots + 500\frac{D_{39}}{D_{20}},$$

$$x \cdot D_{20} = 1000D_{20} + 950D_{21} + 900D_{22} + \cdots + 550D_{29}$$
$$+ 500(D_{30} + D_{31} + D_{32} + \cdots + D_{39})$$

$$= 1000(D_{20} + D_{21} + D_{22} + \cdots + D_{39})$$
$$-(50D_{21} + 100D_{22} + 150D_{23} + \cdots + 450D_{29}$$
$$+ 500D_{30} + 500D_{31} + 500D_{32} + \cdots + 500D_{39})$$

$$= 1000(N_{20} - N_{40}) - 50(D_{21} + 2D_{22} + 3D_{23} + \cdots$$
$$+ 9D_{29}) - 500(N_{30} - N_{40})$$

$$= 1000N_{20} - 500N_{30} - 500N_{40}$$
$$- 50[(D_{21} + D_{22} + D_{23} + \cdots + D_{29})$$
$$+ (D_{22} + D_{23} + \cdots + D_{29})$$
$$+ (D_{23} + \cdots + D_{29}) + \cdots + (D_{29})]$$

$$= 1000N_{20} - 500N_{30} - 500N_{40} - 50[(N_{21} - N_{30})$$
$$+ (N_{22} - N_{30}) + (N_{23} - N_{30}) + \cdots (N_{29} - N_{30})]$$

$$= 1000N_{20} - 500N_{30} - 500N_{40}$$
$$- 50[(N_{21} + N_{22} + N_{23} + \cdots + N_{29}) - 9N_{30}]$$
$$= 1000N_{20} - 500N_{30} - 500N_{40} - 50(S_{21} - S_{30} - 9N_{30})$$

$$= 1000N_{20} - 50N_{30} - 500N_{40} - 50S_{21} + 50S_{30}$$

$$= 1000N_{20} - 500N_{40} - 50S_{21} + 50(S_{30} - N_{30})$$

$$= 1000N_{20} - 500N_{40} - 50S_{21} + 50S_{31}.$$

Hence

$$x = \frac{1000N_{20} - 500N_{40} - 50S_{21} + 50S_{31}}{D_{20}}$$
$$= \$10{,}319.87.$$

Note: With practice, the reader should be able to write down the answer (in commutation symbols) to problems of this type without going through the detailed analysis of either of the solutions presented above. Even after attaining such proficiency, however, the reader should not lose sight of the fact that problems involving varying annuity payments may be worked by going back to first principles. There are two major circumstances under which it is generally advisable to attack a varying annuity problem from first principles:

(*a*) Some problems are so complicated that application of first principles is the most economical method as well as the safest method from the standpoint of being certain that the reasoning is correct.

(*b*) Most persons who are engaged in actuarial work will have occasion to work with varying annuity problems only at intervals. Proficiency acquired in writing down the answer at once is easily lost. In a practical situation, unless a person is confronted with this type of problem frequently, it is probably better to resort to first principles, thus depending on algebraic accuracy rather than on memory or more sophisticated reasoning.

EXAMPLE 3

Describe the annuity whose present value is represented by

$$\frac{S_{40} - S_{60} - 20N_{60}}{D_{20}}.$$

Solution:

$$\begin{aligned}
&S_{40} - S_{60} - 20N_{60} \\
&= (N_{40} + N_{41} + N_{42} + \cdots + N_{59} + N_{60} + N_{61} + \cdots) \\
&\qquad\qquad - (N_{60} + N_{61} + \cdots) - 20N_{60} \\
&= N_{40} + N_{41} + N_{42} + \cdots + N_{59} - 20N_{60} \\
&= D_{40} + D_{41} + D_{42} + \cdots + D_{59} + D_{60} + D_{61} + \cdots \\
&\qquad + D_{41} + D_{42} + \cdots + D_{59} + D_{60} + D_{61} + \cdots \\
&\qquad\qquad + D_{42} + \cdots + D_{59} + D_{60} + D_{61} + \cdots \\
&\qquad\qquad\qquad + \cdots \\
&\qquad\qquad\qquad\qquad + D_{59} + D_{60} + D_{61} + \cdots \\
&\qquad\qquad\qquad\qquad - 20(D_{60} + D_{61} + \cdots) \\
&= D_{40} + 2D_{41} + 3D_{42} + \cdots + 20D_{59}.
\end{aligned}$$

Hence

$$\frac{S_{40} - S_{60} - 20N_{60}}{D_{20}} = \frac{D_{40}}{D_{20}} + 2\frac{D_{41}}{D_{20}} + 3\frac{D_{42}}{D_{20}} + \cdots + 20\frac{D_{59}}{D_{20}}$$

and apparently represents the value to a person aged 20 of a temporary deferred increasing life annuity involving a payment of \$1 at age 40, \$2 at age 41, \$3 at age 42, increasing by \$1 a year until the last payment (of \$20) is due at age 59.

Note: This problem has been analyzed from first principles. With a thorough understanding of the meaning of the commutation symbols and a little practice, the reader should be able to analyze problems of this type without such a detailed analysis. It cannot be emphasized too strongly, however, that in case of doubt resort may always be made to a detailed analysis.

PROBLEM SET 27

1. Prove that

$$D_x + 2D_{x+1} + 3D_{x+2} + \cdots + nD_{x+n-1} = S_x - S_{x+n} - nN_{x+n}.$$

2. Find the present value of an increasing life annuity to (30) where a payment of \$100 is due at age 30, \$200 at age 31, \$300 at age 32, and so on.

3. Find the present value of an increasing life annuity to (30) where a payment of \$100 is due at age 31, \$200 at age 32, \$300 at age 33, and so on.

4. Find the present value of an increasing life annuity to (30) where a payment of \$100 is due at age 40, \$200 at age 41, \$300 at age 42, and so on.

5. Find the present value of a temporary life annuity to (35) where a payment of \$1000 is due at age 36, \$1100 at age 37, \$1200 at age 38, and so on, the last payment being due at age 60.

6. A man now aged 40 has a temporary life annuity with successive annual payments of \$10, \$7, \$4, \$1, \$4, \$7, \$10, the first payment to be made immediately. Compute the present value.

7. Express in commutation symbols the present value to (25) of each of the following series of payments.

(*a*) \$100 at age 26, \$90 at age 27, \$80 at age 28, and so on, until the last payment of \$10 is made.

(*b*) \$1000 at age 30, \$800 at age 31, \$600 at age 32, \$400 at age 33, \$200 every year thereafter, with the last payment due at age 60.

(*c*) \$100 every year from ages 35 to 64, inclusive, \$1100 at age 65, \$200 at age 66, \$300 at age 67, \$400 at age 68, and so on, for life.

(*d*) \$100 at age 25, \$101 at age 26, \$102 at age 27, increasing to a maximum of \$130, after which the payments remain constant.

8. Find the net annual premium for a deferred life annuity issued at age 35 where the first payment of \$100 is due at age 70, the second payment of \$120 is due

at age 71, the third payment of \$140 at age 72, and so on, and where the premiums are limited to 30 years.

9. Describe the annuity whose present value is represented by each of the following:

(a) $\dfrac{S_{35}}{D_{20}}$.

(b) $\dfrac{S_{26} - S_{56}}{D_{25}}$.

(c) $\dfrac{S_{30} - S_{50} - 20N_{50}}{D_{30}}$.

(d) $\dfrac{N_{20} + S_{61} + 10D_{70}}{D_{20}}$.

(e) $\dfrac{hN_x + kS_{x+1}}{D_x}$.

10. Solve problem 3 if (30) is male and mortality follows the 1937 Standard Annuity Table (set back 1 year).

11. Solve problem 10 if (30) is female.

38 • INCREASING INSURANCE

The symbol $(IA)_x$ denotes the present value of an increasing insurance benefit issued at age x under which \$1 is payable at the end of the first year if (x) dies during the first year, \$2 at the end of the second year if (x) dies during the second year, \$3 at the end of the third year if (x) dies during the third year, etc. The benefit is apparently the sum of a series of whole-life benefits, the first to begin right now, the second to begin a year from now, etc. We have

$$(IA)_x = A_x + {}_{1|}A_x + {}_{2|}A_x + \cdots + {}_{\omega-x|}A_x$$
$$= \frac{M_x + M_{x+1} + M_{x+2} + \cdots + M_\omega}{D_x}$$

or

$$(IA)_x = \frac{R_x}{D_x} \tag{57}$$

by the definition in equations 19.

Example 1

A man now aged 30 has a term insurance policy which provides a death benefit of \$1000 in the first year, \$1200 in the second year, \$1400 in the third year, and so on, for 20 years of coverage in all. Find the net single premium.

Solution: The method of attack is very similar to that of example 2, section 37. Letting x be the net single premium, we have

$$x = 1000\frac{C_{30}}{D_{30}} + 1200\frac{C_{31}}{D_{30}} + 1400\frac{C_{32}}{D_{30}} + \cdots + 4800\frac{C_{49}}{D_{30}},$$

$$\begin{aligned}
x \cdot D_{30} &= 1000(C_{30} + C_{31} + C_{32} + \cdots + C_{49}) \\
&\qquad + (200C_{31} + 400C_{32} + \cdots + 3800C_{49}) \\
&= 1000(M_{30} - M_{50}) + 200(C_{31} + 2C_{32} + 3C_{33} + \cdots \\
&\qquad + 19C_{49}) \\
&= 100(M_{30} - M_{50}) + 200[(C_{31} + C_{32} + C_{33} + \cdots + C_{49}) \\
&\qquad + (C_{32} + C_{33} + \cdots + C_{49}) \\
&\qquad + (C_{33} + \cdots + C_{49}) + \cdots + (C_{49})] \\
&= 1000(M_{30} - M_{50}) + 200[(M_{31} - M_{50}) + (M_{32} - M_{50}) \\
&\qquad + (M_{33} - M_{50}) + \cdots + (M_{49} - M_{50})] \\
&= 1000(M_{30} - M_{50}) + 200[(M_{31} + M_{32} + M_{33} + \cdots \\
&\qquad + M_{49}) - 19M_{50}] \\
&= 1000(M_{30} - M_{50}) + 200(R_{31} - R_{50}) - 3800M_{50} \\
&= 1000M_{30} + 200R_{31} - 200R_{50} - 4800M_{50} \\
&= 1000M_{30} - 5000M_{50} + 200(R_{31} - R_{51}).
\end{aligned}$$

Hence,

$$\begin{aligned}
x &= \frac{1000M_{30} - 5000M_{50} + 200(R_{31} - R_{51})}{D_{30}} \\
&= \$284.57.
\end{aligned}$$

EXAMPLE 2

A policy issued at age 10 provides that (*a*) if death occurs before age 20, the death benefit is the return without interest of the gross premiums paid; (*b*) if death occurs after age 20, the death benefit is \$5000; (*c*) premiums shall be paid annually and limited in number to 10; (*d*) the gross premium shall be 20% more than the net. Find the net premium.

Solution: Let P be the net premium. Then the gross premium is $1.2P$, and, equating the present value of future net premiums to the present value of future benefits, we have

$$P\ddot{a}_{10:\overline{10|}} = 1.2P\frac{C_{10}}{D_{10}} + 2.4P\frac{C_{11}}{D_{10}} + 3.6P\frac{C_{12}}{D_{10}} + \cdots + 12P\frac{C_{19}}{D_{10}} + 5000\frac{M_{20}}{D_{10}},$$

$$P(N_{10} - N_{20}) = 1.2P(C_{10} + 2C_{11} + 3C_{12} + \cdots + 10C_{19}) + 5000M_{20}$$

$$= 1.2P(M_{10} + M_{11} + M_{12} + \cdots + M_{19} - 10M_{20}) + 5000M_{20}$$

$$= 1.2P(R_{10} - R_{20} - 10M_{20}) + 5000M_{20},$$

$$P(N_{10} - N_{20} - 1.2R_{10} + 1.2R_{20} + 12M_{20}) = 5000M_{20},$$

$$P = \frac{5000M_{20}}{N_{10} - N_{20} - 1.2R_{10} + 1.2R_{20} + 12M_{20}}$$

$$= \$216.72.$$

Note: Examples 1 and 2 above have been worked from first principles. The comments made in connection with examples 2 and 3 of section 37 apply with equal force here.

PROBLEM SET 28

1. (*a*) Prove that

$$C_x + 2C_{x+1} + 3C_{x+2} + \cdots + nC_{x+n-1} = R_x - R_{x+n} - nM_{x+n}.$$

(*b*) By analogy with example 1 of section 37, define $(IA)^{1}_{x:\overline{n}|}$ and express its value in terms of commutation symbols.

2. A man now aged 30 has a whole-life insurance policy which provides a death benefit of \$10,000 in the first year, \$9700 in the second year, \$9400 in the third year, decreasing by \$300 each year until the benefit is \$4000, at which time the benefit remains constant. Find the net single premium.

3. A 10-pay life policy issued at age 8 provides that, if death occurs before age 18, the net premiums without interest will be returned and the face amount will be paid; if death occurs after age 18, the face amount only will be paid. Find the net annual premium for a \$1000 policy.

4. The gross single premium A' for a whole-life policy which provides for a payment at death of the gross premium plus \$1000 is related to the net single premium A by the equation

$$0.9A' = A + 25.$$

Calculate the gross single premium for age 20.

5. A life insurance policy issued at age 20 provides for 20 annual premiums. If the insured dies between ages 20 and 30, the death benefit is \$1000. If he dies between 30 and 50, the death benefit is \$3000. If he dies between 50 and 70, the death benefit is \$2000. If he survives to age 70, the policy pays \$1000. Express the net annual premium in commutation symbols.

6. A child's endowment policy issued at age 1 provides for a death benefit of \$100 in the first year, \$200 the second year, \$300 the third year, increasing every year until a maximum of \$1000 is reached. The policy matures at age 18 with an endowment of \$1000. Express the net annual premium in commutation symbols.

7. A 20-pay 30-year endowment policy with a face amount of \$1000 provides that in the event of death during the 30 years the net premiums paid will be

refunded along with the payment of the face amount. Express the net annual premium in terms of commutation symbols.

★ **8.** A family income policy provides that, if (30) dies before age 49, a payment of \$100 will be made at the end of the year of death and further payments of \$100 will be made every month until the last one is made 19 years and 11 months after the issue date of the policy. Furthermore, if (30) dies between ages 30 and 50, \$10,000 will be paid 20 years after the date of issue. The policy expires at age 50. Find an expression for the net single premium.

9. Describe the type of insurance whose net single premium is given by each of the following:

$$(a)\quad \frac{1000(M_x + 2R_{x+1})}{D_x}.$$

$$(b)\quad \frac{1000(M_x - M_{x+10} + 2D_{x+10})}{D_x}.$$

$$(c)\quad \frac{1000(R_{x+5} - R_{65})}{D_x}.$$

$$(d)\quad \frac{1000M_x}{D_x - M_x}.$$

10. Describe the type of policy whose net annual premium is given by each of the following:

$$(a)\quad \frac{1000(M_{30} + 2R_{31})}{N_{30} - N_{50}}.$$

$$(b)\quad \frac{1000R_{20}}{N_{20}}.$$

$$(c)\quad \frac{1000(M_{30} - M_{50})}{N_{30}}.$$

$$(d)\quad \frac{1000(M_x - M_{x+20} + D_{x+20})}{N_x - N_{x+20} - R_x + R_{x+20} + 20M_{x+20}}.$$

11. Discuss the policy represented in problem 10*c* from the standpoint of whether it would be practical for issue by a life insurance company. Practical considerations aside, would such a policy be equitable?

12. A whole-life policy provides for an increasing death benefit under which the death benefit in the *n*th year is $(1.01)^n$. If the company calculates net premiums on a $2\frac{1}{2}\%$ interest assumption, the net annual premium for the policy is $A_x/\ddot{a}_x$, where $\ddot{a}_x$ is computed at $2\frac{1}{2}\%$ and A_x at rate i. Find i.

PROBLEM SET 29

1. Evaluate $100\ddot{a}_{60}^{(12)}$.

2. Describe in words the annuity represented by

$$3000\ddot{a}_{\overline{10}|}^{(12)} + 3000 \cdot {}_{10|}\ddot{a}_{55}^{(12)}$$

and evaluate it.

3. Find q_x if $i = 0.04$ and $\ddot{a}_{x:\overline{1}|}^{(12)} = 0.95417$.

4. Describe the annuity whose present value is represented by $(S_{40} - S_{50})/D_{18}$.

5. Calculate the net monthly premium for a $1000 ordinary life policy issued at age 20.

6. Calculate the net monthly premium for a $1000 20-pay life policy issued at age 18.

7. A man now aged 25 has a whole-life insurance policy which provides a death benefit of $10,000 in the first year, $9500 in the second year, $9000 in the third year, decreasing by $500 each year until the benefit is $5000, at which time the benefit remains constant. Find the net single premium.

8. A life insurance policy is issued to a man aged 45 providing that, if the insured dies before age 65, his beneficiary will receive $100 a month for 20 years, the first payment to be made at the end of the policy year in which the insured dies. If premiums are limited to a maximum of 10 years, calculate the net annual premium.

9. A man now aged 28 has a whole-life insurance policy which provides a death benefit of $10,000 during the first 20 years, $8000 during the next 17 years, and $5000 thereafter. Premiums are limited to 32 years. Calculate the tenth mean reserve.

10. Prove by general reasoning that $\ddot{a}_{\overline{n}|}^{(12)} < \ddot{a}_{\overline{n}|}$.

CHAPTER SEVEN

Modified Reserves

39 · INTRODUCTION

The premium actually charged for life insurance may be broken down into two parts, the net premium and the loading. We have the relation

$$\text{Gross premium} = \text{Net premium} + \text{Loading}.$$

The net premium would pay for the insurance benefits exactly if:

(*a*) The interest earned on the insurance company's funds is exactly at the rate assumed in calculating the net premium.

(*b*) The company's mortality experience is exactly the same as the mortality table used in the calculations says it will be.

(*c*) The insurance company is able to administer the policy without incurring any expense.

(*d*) Death benefits are paid at the end of the policy year in which death occurs.

Calculating reserves by the net level premium method is consistent with these assumptions and makes the further assumption that the loading is just sufficient to cover the expenses. This further assumption implies that, since the loading is the same every year, the expenses are the same. This assumption is not in accordance with the facts. As insurance is sold in the United States, the first-year expenses are considerably greater than renewal expenses partly because the administrative expenses in putting a new policy on the books are greater than in keeping it there, and partly because first-year commissions to the soliciting agent are almost always higher than the renewal commissions. In fact, first-year expenses usually exceed the loading. This means that an insurance company will not have enough actual money left from the first-year premium, after paying expenses and death claims, to balance the net level reserve. Thus, the issuing of a new policy will actually decrease the surplus of the company, where the surplus is

defined as the assets minus the liabilities. In life insurance accounting the policy reserves are the lion's share of the liabilities.

The drain on surplus caused by new business is not serious for an old company whose ratio of new business to old is relatively small. In the second and subsequent years the loading is ordinarily more than enough to cover the renewal expenses so that the old policies will increase surplus by more than enough to balance the new business drain. The picture is different for a young company, or for an old company whose ratio of new business to old is relatively large. In such a company setting up full net level reserves may produce a situation in which the new business deficit exceeds the gain from old business. In this case the company's surplus would decrease or possibly even become negative so that a superficial comparison of assets and liabilities would indicate that the company was insolvent.

Since requiring full net level reserves would make it practically impossible to operate a new life insurance company and would put artificial brakes on a rapidly growing company, various modified reserve systems have been approved by law and put to actual use by life insurance companies. Such systems are financially sound and, in fact, are more realistic than the full net level method. A company that values its policy liabilities on the full net level basis is not necessarily stronger financially than a company that uses a modified reserve system.

In practically all modified reserve systems the valuation net premium for the first year, called α, is less than the valuation net premium for the renewal years, called β. The present value, at date of issue, of the valuation net premiums is equal to the present value of the net level premiums. Under an n-pay policy issued to (x) we have the following relation:

$$\alpha + \beta a_{x:\overline{n-1|}} = P\ddot{a}_{x:\overline{n|}}. \tag{58}$$

Since $\alpha < \beta$, it follows that $\alpha < P < \beta$; that is, the first-year modified valuation premium is less than the net level premium, which is less than the renewal modified valuation premium.

40 · FULL PRELIMINARY TERM

Under the full preliminary term method a policy is considered mathematically as if it were a combination of 1-year term insurance and a 1-year deferred permanent plan. For example a 20-year endowment issued at age 25 would be considered for valuation purposes as a 1-year term policy at age 25 plus a 19-year endowment issued at age 25 but deferred to age 26. The α for the full preliminary term method may be denoted by α_F and is just enough to pay for the first-year death

claims with the mortality and interest assumptions used in calculating P and in accumulating the reserves. Therefore, for a \$1 policy,

$$\alpha_F = c_x$$

and

$$\alpha_F + \beta_F a_{x:\overline{n-1|}} = P\ddot{a}_{x:\overline{n|}}$$

so that

$$\beta_F = \frac{P\ddot{a}_{x:\overline{n|}} - \alpha_F}{a_{x:\overline{n-1|}}}$$

$$= \frac{P(1 + a_{x:\overline{n-1|}}) - \alpha_F}{a_{x:\overline{n-1|}}}$$

or

$$\beta_F = P + \frac{P - \alpha_F}{a_{x:\overline{n-1|}}}. \tag{59}$$

Equation 59 is subject to an interesting verbal interpretation. Because of heavy first-year expenses, an amount equal to $P - \alpha_F$ was borrowed from the net premium to give a first-year expense allowance of $P - \alpha_F$ plus the loading. This loan is to be amortized over the remaining premium paying period by paying in P plus enough extra to amortize the loan. The extra amount must be $(P - \alpha_F)/a_{x:\overline{n-1|}}$ because the present value of a series of payments of that size is $P - \alpha_F$, the amount of the loan.

Example

Calculate the full preliminary term reserves for the first 5 years for a \$1000 20-pay, 30-year endowment issued at age 30. Compare the results with the full net level reserves calculated in the example of section 33.

Solution: From section 33, $P = 33.7900$. Since $\alpha_F = 1000c_{30} = 3.47359$, we have from formula 59

$$\beta_F = 33.7900 + \frac{33.7900 - 3.4736}{a_{30:\overline{19|}}}$$

$$= 33.7900 + \frac{30.3164}{14.301234}$$

$$= 33.7900 + 2.1198$$

$$= 35.9098.$$

Using Fackler's method, we have

$$
\begin{aligned}
{}_1V^F &= \alpha_F u_{30} - 1000k_{30} \\
&= (3.4736)(1.0286625) - 3.57315 \\
&= 0. \\
{}_2V^F &= ({}_1V^F + \beta_F)u_{31} - 1000k_{31} \\
&= (35.9098)(1.0288381) - 3.74450 \\
&= 33.2009. \\
{}_3V^F &= ({}_2V^F + \beta_F)u_{32} - 1000k_{32} \\
&= (69.1107)(1.0290337) - 3.93533 \\
&= 67.1819. \\
{}_4V^F &= ({}_3V^F + \beta_F)u_{33} - 1000k_{33} \\
&= (103.0917)(1.0292407) - 4.13722 \\
&= 101.9690. \\
{}_5V^F &= ({}_4V^F + \beta_F)u_{34} - 1000k_{34} \\
&= (137.8788)(1.0294785) - 4.36929 \\
&= 137.5740.
\end{aligned}
$$

Check:

$$
\begin{aligned}
{}_5V^F &= 1000A_{35:\overline{25|}} - \beta_F \ddot{a}_{35:\overline{15|}} \\
&= 575.73071 - (35.9098)(12.201594) \\
&= 575.73071 - 438.15680 \\
&= 137.5739.
\end{aligned}
$$

The following table compares the full preliminary term and full net level reserves.

Duration	*FNL*	*FPT*	*Difference*
1	\$ 31.19	\$ 0	\$31.19
2	63.10	33.20	29.90
3	95.77	67.18	28.59
4	129.21	101.97	27.24
5	163.44	137.57	25.87

If the tables were continued, it would turn out that ${}_{20}V^L = {}_{20}V^F$. Full net level reserves are always equal to full preliminary term reserves at the end of the premium paying period.

41 • COMMISSIONERS RESERVE VALUATION METHOD

There is a serious objection to applying the full preliminary term method to all policies. The extra expense allowance is much higher than necessary for the higher premium policies. Consider the following table, which shows the extra allowance ($P - \alpha_F$) for several different plans of insurance issued at age 40 for a face amount of \$1000.

Plan	P	$P - \alpha_F$
Ordinary life	\$24.65	\$18.62
Endowment at 85	24.92	18.89
Paid-up at 65	29.87	23.84
10-pay life	57.84	51.81
10-year endowment	90.68	84.65

It seems safe to conclude from this table that, if \$18.62 is a reasonable expense allowance for ordinary life, \$84.65 is unreasonably liberal for 10-year endowment. A modified system which allows FPT for the lower premium policies but requires a reserve somewhere between FPT and FNL for the higher premium policies would seem to be called for.

In December 1942 the Standard Nonforfeiture and Valuation Laws were unanimously approved by the National Association of Insurance Commissioners. These laws, popularly called the Guertin* Laws, have at the present writing been enacted by most of the states. Section 4 of the Standard Valuation Law defines the Commissioners reserve valuation method. Under the Commissioners method the reserve for a \$1 policy is defined as the present value of future benefits less the present value of future "modified net premiums," where the modified net premium, β_C, satisfies the following equation:

$$\beta_C \ddot{a}_{x:\overline{n}|} = (\text{NSP}) + [(a) - (b)]$$

where x = age at issue,
n = number of annual premiums required by the policy,
NSP = net single premium for the benefit,
$(a) = \beta_F$ or ${}_{19}P_{x+1}$, whichever is smaller,
$(b) = c_x$,

and negative values† of $[(a) - (b)]$ are taken as 0.

* Alfred N. Guertin was Chairman of the Committee to Study the Need for a New Mortality Table and Related Topics, a committee of the National Association of Insurance Commissioners. At the time he served as chairman of this committee, Mr. Guertin was Actuary for the State of New Jersey.

† $[(a) - (b)]$ is negative only in very special cases. Such cases are beyond the scope of this book.

Since $\text{NSP}/\ddot{a}_{x:\overline{n}|} = P$, we have

$$\beta_C = P + \frac{[(a) - (b)]}{\ddot{a}_{x:\overline{n}|}}. \tag{60}$$

From equation 58,

$$\begin{aligned} \alpha_C + \beta_C a_{x:\overline{n-1}|} &= P\ddot{a}_{x:\overline{n}|}, \\ \alpha_C + \beta_C(\ddot{a}_{x:\overline{n}|} - 1) &= P\ddot{a}_{x:\overline{n}|}, \\ \beta_C - \alpha_C &= (\beta_C - P)\ddot{a}_{x:\overline{n}|} \\ &= [(a) - (b)] \end{aligned}$$

or

$$\alpha_C = \beta_C - [(a) - (b)]. \tag{61}$$

$[(a) - (b)]$ is called the "excess." Values of α_C and β_C for the more common types of policies have been published in a series of volumes by the Actuarial Society of America and the American Institute of Actuaries. Incidentally, these two bodies merged in 1949 into one body called the Society of Actuaries.

Example

Calculate the Commissioners method reserves for the first 5 years for a \$1000 20-pay, 30-year endowment issued at age 30. Compare the results with the FNL and FPT reserves shown in section 40.

Solution: From section 40, $P = 33.7900$ and $\beta_F = 35.9098$.

$$\begin{aligned} 1000 \cdot {}_{19}P_{31} &= \frac{1000M_{31}}{N_{31} - N_{50}} \\ &= \frac{180{,}872{,}340}{6{,}303{,}992} \\ &= 28.6917. \end{aligned}$$

Since $\beta_F > 1000 \cdot {}_{19}P_{31}$, $(a) = 1000 \cdot {}_{19}P_{31} = 28.6917$. $(b) = 1000c_{30} = 3.4736$.

From equation 60,

$$\begin{aligned} \beta_C &= 33.7900 + \frac{28.6917 - 3.4736}{\ddot{a}_{30:\overline{20}|}} \\ &= 33.7900 + 1.6481 \\ &= 35.4381. \end{aligned}$$

From equation 61,

$$\alpha_C = 35.4381 - (28.6917 - 3.4736)$$
$$= 10.2200.$$

Using Fackler's method,

$$_1V^C = \alpha_c u_{30} - 1000k_{30}$$
$$= (10.2200)(1.0286625) - 3.57315$$
$$= 6.9398.$$
$$_2V^C = ({}_1V^C + \beta_C)u_{31} - 1000k_{31}$$
$$= (42.3779)(1.0288381) - 3.74450$$
$$= 39.8555.$$
$$_3V^C = ({}_2V^C + \beta_c)u_{32} - 1000k_{32}$$
$$= (75.2936)(1.0290337) - 3.93533$$
$$= 73.5443.$$
$$_4V^C = ({}_3V^C + \beta_c)u_{33} - 1000k_{33}$$
$$= (108.9824)(1.0292407) - 4.13722$$
$$= 108.0319.$$
$$_5V^C = ({}_4V^C + \beta_C)u_{34} - 1000k_{34}$$
$$= (143.4700)(1.0294785) - 4.36929$$
$$= 143.3300.$$

Check:

$$_5V^C = 1000A_{35:\overline{25|}} - \beta_C\ddot{a}_{35:\overline{15|}}$$
$$= 575.73071 - (35.4381)(12.201594)$$
$$= 575.73071 - 432.40131$$
$$= 143.3294.$$

The following table compares the full net level, Commissioners method, and full preliminary term reserves.

Duration	*FNL*	*CVM*	*FPT*
1	\$ 31.19	\$ 6.94	\$ 0
2	63.10	39.86	33.20
3	95.77	73.54	67.18
4	129.21	108.03	101.97
5	163.44	143.33	137.57

If the table were continued it would turn out that ${}_{20}V^L = {}_{20}V^C = {}_{20}V^F$. Full net level reserves, Commissioners method reserves, and full preliminary term reserves are always all equal at the end of the premium paying period.

PROBLEM SET 30

1. Calculate the first five terminal reserves for a \$1000 20-year endowment policy issued at age 25 on the FNL, CVM, and FPT bases. Check by using the prospective method.

2. Calculate the terminal reserves for a \$1000 5-year endowment policy issued at age 50 on the FNL, CVM, and FPT bases.

3. Calculate the terminal reserves for a \$1000 5-year term policy issued at age 60 on the FNL, CVM, and FPT bases.

4. Calculate the first and third mean reserves on each of the three bases for the policy in problem 1.

5. Calculate α_F and β_F for the following \$1000 policies issued at age 23.

(*a*) Ordinary life.
(*b*) Life paid-up at 65.
(*c*) Life paid-up at 60.
(*d*) 20-pay life.
(*e*) 10-pay life.
(*f*) 20-year term.
(*g*) Term to age 65.
(*h*) Endowment at age 85.
(*i*) Endowment at age 65.
(*j*) 20-year endowment.
(*k*) 10-year endowment.

6. Calculate α_C and β_C for the policies in problem 5.

42 • OTHER MODIFIED RESERVE METHODS

Prior to the adoption of the Standard Valuation Laws various other modified reserve methods have been in use. Although these older methods are not used for currently issued policies, they are of more than academic interest because they are still used to value policy liabilities for older policies. Five of these methods will be considered in this chapter. They are: the Ohio method; the Canadian method; the Illinois method; the New Jersey method; and the Select and Ultimate method.

Of these the Illinois method is the most important, having been incorporated into the insurance laws of a number of states. As will be seen in sections 45 and 48, the Illinois method is very similar to the Commissioners method. The essential difference is that the Commissioners method brings the reserve up to full net level at the end of the premium paying period, whereas the Illinois method brings the reserve up to full net level at the end of the premium paying period or at the end of 20 years, whichever comes first. Thus the Illinois method produces somewhat higher aggregate reserves than does the Commissioners method, assuming the same mortality table and rate of interest.

It is important to distinguish between a *valuation method* and a *valuation standard*.

Essentially, a valuation method consists of a set of specific and consistent actuarial rules by which a set of reserves can be calculated. A valuation standard is a legal requirement of a state (or some other political unit) defining the particular actuarial rules to be used with a specified interest rate and mortality table in order to produce the minimum reserve requirements for specified groups of policies. Most valuation standards involve more than one valuation method. For example, the Illinois standard defines two groups of policies, using the Illinois method for one group and the FPT method for the other.

With the Commissioners method, this distinction is not necessary because the method and the standard amount to the same thing.

43 • THE OHIO METHOD

Under the Ohio standard, policies are divided into two groups:

(*a*) Limited payment life or endowment policies providing for premium payment periods of less than 20 years.

(*b*) Other policies.

The standard provides that the Ohio method (sometimes called the ordinary life modification) will be used for policies in group 1 and that the FPT method will be used for policies in group 2.

Under the Ohio method, the reserve is the FPT reserve for an ordinary life policy with the same age at issue plus an amount equal to the accumulation of a net level premium sufficient to provide for a pure endowment at the end of the premium paying period equal to the difference at the end of that period between the FPT reserve for the ordinary life policy and the FNL reserve for the policy in question. In symhols, the method states that

$$_tV^O = {}_{t-1}V_{x+1} + \pi_O \cdot {}_tu_x$$

and

$$\pi_O \cdot {}_nu_x = {}_nV^L - {}_{n-1}V_{x+1}.$$

Alternatively, it can be deduced from the verbal description of the method that π_O must be added to the ordinary life FPT valuation net premiums in order to produce the Ohio method valuation net premiums. That is,

$$\alpha_O = c_x + \pi_O,$$

$$\beta_O = P_{x+1} + \pi_O.$$

Subtracting, we have

$$\beta_O - \alpha_O = P_{x+1} - c_x$$

or

$$\alpha_O = \beta_O - (P_{x+1} - c_x). \tag{62}$$

Applying equation 58, we have

$$\alpha_O + \beta_O a_{x:\overline{n-1}|} = P\ddot{a}_{x:\overline{n}|}.$$

Substituting for α_O from equation 62, we have

$$\beta_O - (P_{x+1} - c_x) + \beta_O a_{x:\overline{n-1}|} = P\ddot{a}_{x:\overline{n}|},$$

$$\beta_O \ddot{a}_{x:\overline{n}|} - (P_{x+1} - c_x) = P\ddot{a}_{x:\overline{n}|},$$

or

$$\beta_O = P + \frac{P_{x+1} - c_x}{\ddot{a}_{x:\overline{n}|}}. \tag{63}$$

Example

Calculate the first five terminal reserves under the Ohio standard for a $1000 15-pay life policy issued at age 30.

Solution: Since the policy is in group 1, the Ohio method must be used.

$$P = 1000\,\frac{A_{30}}{\ddot{a}_{30:\overline{15}|}}$$

$$= \frac{413.80049}{12.323678}$$

$$= 33.5777.$$

$$\frac{1000(P_{31} - c_{30})}{\ddot{a}_{30:\overline{15}|}} = \frac{17.81383 - 3.47359}{12.323678}$$

$$= \frac{14.34024}{12.323678}$$

$$= 1.1636.$$

From equation 63,

$$\beta_O = 33.5777 + 1.1636$$

$$= 34.7413.$$

From equation 62,

$$\alpha_O = 34.7413 - 14.3402$$

$$= 20.4011.$$

Using Fackler's method,

$${}_1V^O = \alpha_O u_{30} - 1000k_{30}$$

$$= 17.4127.$$

$${}_2V^O = ({}_1V^O + \beta_O)u_{31} - 1000k_{31}$$

$$= 49.9135.$$

$$
\begin{aligned}
{}_3V^O &= ({}_2V^O + \beta_O)u_{32} - 1000k_{32} \\
&= 83.1773. \\
{}_4V^O &= ({}_3V^O + \beta_O)u_{33} - 1000k_{33} \\
&= 117.2294. \\
{}_5V^O &= ({}_4V^O + \beta_O)u_{34} - 1000k_{34} \\
&= 152.0813.
\end{aligned}
$$

Check:

$$
\begin{aligned}
{}_5V^O &= 1000A_{35} - \beta_O\ddot{a}_{35:\overline{10}|} \\
&= 456.6121 - (34.7413)(8.765669) \\
&= 152.0814.
\end{aligned}
$$

PROBLEM SET 31

1. Calculate the first five terminal reserves under the Ohio standard for a \$1000 15-year endowment policy issued at age 25.

2. Calculate the first five mean reserves for the policy in the example in section 43.

3. Calculate the first mean reserves for the policy in problem 1.

4. Calculate α and β under the Ohio standard for a \$1000 policy issued at age 25 under each of the following plans:

(*a*) Ordinary life.
(*b*) 30-year endowment.
(*c*) 10-year endowment.
(*d*) 10-pay life.
(*e*) 20-pay life.
(*f*) Term to age 65.
(*g*) 10-year term.

44 · THE CANADIAN METHOD

Under the Canadian standard, policies are divided into two groups:

(*a*) Policies with net level premiums greater than the corresponding net level ordinary life premium at the same age at issue.

(*b*) Other policies.

The standard provides that the Canadian method will be used for policies in group 1 and that the FPT method will be used for policies in group 2.

Under the Canadian method, the *t*th reserve is the value at age $x + t$ of future benefits minus the value at age $x + t$ of future valuation net premiums, where the valuation net premium is defined as

$$\beta_D = P + \frac{P_x - c_x}{a_{x:\overline{n-1}|}}.* \tag{64}$$

* The D in β_D stands for Dominion. C has already been used for the Commissioners method.

At the end of the premium paying period, the value of future valuation net premiums is 0 and the Canadian method reserve is the same as FNL. Therefore, equation 58 applies, and

$$\alpha_D + \beta_D a_{x:\overline{n-1}|} = P\ddot{a}_{x:\overline{n}|}.$$

Hence

$$\begin{aligned}\alpha_D &= P\ddot{a}_{x:\overline{n}|} - \beta_D a_{x:\overline{n-1}|} \\ &= P\ddot{a}_{x:\overline{n}|} - \left(P + \frac{P_x - c_x}{a_{x:\overline{n-1}|}}\right) a_{x:\overline{n-1}|} \\ &= P\ddot{a}_{x:\overline{n}|} - Pa_{x:\overline{n-1}|} - (P_x - c_x),\end{aligned}$$

or

$$\alpha_D = P - (P_x - c_x). \tag{65}$$

Example

Calculate the first five terminal reserves under the Canadian standard for a \$1000 30-pay life policy issued at age 30.

Solution:

$$\begin{aligned}P &= \frac{1000A_{30}}{\ddot{a}_{30:\overline{30}|}} \\ &= 20.8971. \\ 1000P_{30} &= 17.2172.\end{aligned}$$

Since $P > 1000P_{30}$, the policy is in group 1 and the Canadian method applies.

From equation 65,

$$\begin{aligned}\alpha_D &= P - 1000(P_{30} - c_{30}) \\ &= 20.8971 - (17.2172 - 3.4736) \\ &= 20.8971 - 13.7436 \\ &= 7.1535.\end{aligned}$$

From equation 64,

$$\begin{aligned}\beta_D &= 20.8971 + \frac{13.7436}{a_{30:\overline{29}|}} \\ &= 20.8971 + \frac{13.7436}{18.801850} \\ &= 21.6281.\end{aligned}$$

Applying Fackler's method,

$$ {}_1V^D = \alpha_D u_{30} - 1000k_{30} $$
$$ = 3.7854. $$
$$ {}_2V^D = ({}_1V^D + \beta_D)u_{31} - 1000k_{31} $$
$$ = 22.4019. $$
$$ {}_3V^D = ({}_2V^D + \beta_D)u_{32} - 1000k_{32} $$
$$ = 41.3730. $$
$$ {}_4V^D = ({}_3V^D + \beta_D)u_{33} - 1000k_{33} $$
$$ = 60.7061. $$
$$ {}_5V^D = ({}_4V^D + \beta_D)u_{34} - 1000k_{34} $$
$$ = 80.3920. $$

Check:

$$ {}_5V^D = 1000A_{35} - \beta_D \ddot{a}_{35:\overline{25}|} $$
$$ = 80.3904. $$

PROBLEM SET 32

1. Calculate the first five terminal reserves under the Canadian standard for a $1000 20-pay life policy issued at age 25.

2. Calculate the first five mean reserves for the policy in the example in section 44.

3. Calculate the first five mean reserves for the policy in problem 1.

4. Calculate α and β under the Canadian standard for a $1000 policy issued at age 35 under each of the following plans:

(*a*) Ordinary life
(*b*) 20-year endowment.
(*c*) Endowment at 85.
(*d*) 15-pay life.
(*e*) 20-pay life.
(*f*) Term to age 65.
(*g*) 10-year term.

45 · THE ILLINOIS METHOD

Under the Illinois standard, policies are divided into two groups:

(*a*) Policies with net level premiums greater than the corresponding net level 20-pay life premium at the same age at issue.
(*b*) Other policies.

The standard provides that the Illinois method (sometimes called the 20-payment life modification) will be used for the policies in group 1 and that the FPT method will be used for policies in group 2.

Under the Illinois method, the reserve is the FPT reserve for a

20-pay life policy with the same age at issue plus a certain amount. This amount is the accumulation of a net level premium, π_I. This net level premium is determined in such a way that the reserve under the Illinois method is equal to the FNL reserve at the end of the premium paying period or at the end of 20 years, whichever comes first. In other words, the accumulation of π_I provides for a pure endowment at the end of the premium paying period or at the end of 20 years, whichever comes first, equal to the difference between the FNL reserve for the policy in question and the FPT reserve for the 20-pay life policy. In symbols,

$$_tV^I = {}_{t-1:19}V_{x+1} + \pi_I \cdot {}_tu_x$$

and

$$\pi_I \cdot {}_ku_x = {}_kV^L - {}_{k-1:19}V_{x+1}$$

where k is the smaller of the two quantities, n and 20. Alternatively,

$$\alpha_I = c_x + \pi_I,$$

$$\beta_I = {}_{19}P_{x+1} + \pi_I,$$

$$\beta_I - \alpha_I = {}_{19}P_{x+1} - c_x,$$

or

$$\alpha_I = \beta_I - ({}_{19}P_{x+1} - c_x). \tag{66}$$

For policies having a premium payment period greater than 20 years the Illinois method is noticeably different from the FPT, Commissioners, Ohio, and Canadian methods. Under the latter four methods there are two valuation net premiums, α and β, and the FNL reserve is reached at the end of the premium paying period. Equation 58 holds without adjustment for these methods. Under the Illinois method, there are really three valuation net premiums: α_I for the first year, β_I for the next 19 years, and P thereafter. The FNL reserve is reached at the end of 20 years. Thus, equation 58 must be extended and becomes

$$\alpha_I + \beta_I a_{x:\overline{19|}} + P \cdot {}_{20|n-20}\ddot{a}_x = P\ddot{a}_{x:\overline{n|}}.$$

Since $\ddot{a}_{x:\overline{n|}} = \ddot{a}_{x:\overline{20|}} + {}_{20|n-20}\ddot{a}_x$, the above equation reduces to

$$\alpha_I + \beta_I a_{x:\overline{19|}} = P\ddot{a}_{x:\overline{20|}}.$$

For policies having a premium payment period of 20 years or less, equation 58 holds directly for the Illinois method. Therefore, the general form of equation 58 for the Illinois method is

$$\alpha_I + \beta_I a_{x:\overline{k-1|}} = P\ddot{a}_{x:\overline{k|}} \tag{67}$$

where k is the smaller of the two quantities, n and 20. n, of course, is the premium payment period.

Substituting in equation 67 the value of α_I in equation 66, we have

$$\beta_I - ({}_{19}P_{x+1} - c_x) + \beta_I a_{x:\overline{k-1}|} = P\ddot{a}_{x:\overline{k}|},$$

$$\beta_I \ddot{a}_{x:\overline{k}|} = P\ddot{a}_{x:\overline{k}|} + ({}_{19}P_{x+1} - c_x),$$

or

$$\beta_I = P + \frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{k}|}}. \tag{68}$$

Example 1

Calculate the first five terminal reserves under the Illinois standard for a \$1000 25-year endowment policy issued at age 40.

Solution:

$$P = \frac{1000A_{40:\overline{25}|}}{\ddot{a}_{40:\overline{25}|}}$$

$$= 35.0312,$$

$$1000 \cdot {}_{20}P_{40} = 34.1444.$$

Since $P > 1000 \cdot {}_{20}P_{40}$, the policy is in group 1 and the Illinois method applies.

$$1000 \cdot {}_{19}P_{41} = \frac{1000A_{41}}{\ddot{a}_{41:\overline{19}|}}$$

$$= 36.1935,$$

$$1000c_{40} = 6.0292.$$

Since $n = 25 > 20$, $k = 20$. From equation 68,

$$\beta_I = 35.0312 + \frac{36.1935 - 6.0292}{\ddot{a}_{40:\overline{20}|}}$$

$$= 35.0312 + \frac{30.1643}{14.720974}$$

$$= 35.0312 + 2.0491$$

$$= 37.0803.$$

From equation 66,

$$\alpha_I = 37.0803 - 30.1643$$

$$= 6.9160.$$

Applying Fackler's method,

$$
\begin{aligned}
{}_1V^I &= \alpha_I u_{40} - 1000k_{40} \\
&= 0.9146. \\
{}_2V^I &= ({}_1V^I + \beta_I)u_{41} - 1000k_{41} \\
&= 32.5697. \\
{}_3V^I &= ({}_2V^I + \beta_1)u_{42} - 1000k_{42} \\
&= 64.8167. \\
{}_4V^I &= ({}_3V^I + \beta_I)u_{43} - 1000k_{43} \\
&= 97.6683. \\
{}_5V^I &= ({}_4V^I + \beta_I)u_{44} - 1000k_{44} \\
&= 131.1317.
\end{aligned}
$$

Check:

$$
\begin{aligned}
{}_5V^I &= 1000A_{45:\overline{20}|} - \text{Present value (at age 45) of future valuation net premiums} \\
&= 1000A_{45:\overline{20}|} - \beta_I\ddot{a}_{45:\overline{15}|} - P_{15|5}\ddot{a}_{45} \\
&= 1000A_{45:\overline{20}|} - \beta_I\ddot{a}_{45:\overline{15}|} - P(\ddot{a}_{45:\overline{20}|} - \ddot{a}_{45:\overline{15}|}) \\
&= 1000A_{45:\overline{20}|} - (\beta_I - P)\ddot{a}_{45:\overline{15}|} - P\ddot{a}_{45:\overline{20}|} \\
&= 653.2420 - (37.0803 - 35.0312)(11.745990) \\
&\qquad - (35.0312)(14.217079) \\
&= 131.1320.
\end{aligned}
$$

Example 2

Calculate the first five terminal reserves under the Illinois standard for a \$1000 15-year endowment policy issued at age 20.

Solution:

$$
\begin{aligned}
P &= \frac{1000A_{20:\overline{15}|}}{\ddot{a}_{20:\overline{15}|}} \\
&= 55.8808, \\
1000 \cdot {}_{20}P_{20} &= 21.7646.
\end{aligned}
$$

Since $P > 1000 \cdot {}_{20}P_{20}$, the policy is in group 1 and the Illinois method applies.

$$1000 \cdot {}_{19}P_{21} = \frac{1000A_{21}}{\ddot{a}_{21:\overline{19}|}}$$

$$= 23.0965,$$

$$1000c_{20} = 2.3706,$$

$$1000({}_{19}P_{21} - c_{20}) = 20.7259.$$

Since $n = 15 < 20$, $k = 15$. From equation 68,

$$\beta_I = 55.8808 + \frac{20.7259}{\ddot{a}_{20:\overline{15}|}}$$

$$= 55.8808 + 1.6637$$

$$= 57.5445.$$

From equation 66,

$$\alpha_I = 57.5445 - 20.7259$$

$$= 36.8186.$$

Applying Fackler's method,

$${}_1V^I = \alpha_I u_{20} - 1000k_{20}$$

$$= 35.3952.$$

$${}_2V^I = ({}_1V^I + \beta_I)u_{21} - 1000k_{21}$$

$$= 92.9870.$$

$${}_3V^I = ({}_2V^I + \beta_I)u_{22} - 1000k_{22}$$

$$= 152.0989.$$

$${}_4V^I = ({}_3V^I + \beta_I)u_{23} - 1000k_{23}$$

$$= 212.7746.$$

$${}_5V^I = ({}_4V^I + \beta_I)u_{24} - 1000k_{24}$$

$$= 275.0689.$$

Check:

$${}_5V^I = 1000A_{25:\overline{10}|} - \beta_I\ddot{a}_{25:\overline{10}|}$$

$$= 275.0685.$$

PROBLEM SET 33

1. Calculate the first five terminal reserves under the Illinois standard for a \$1000 15-pay life policy issued at age 25.

2. Calculate the first five terminal reserves under the Illinois standard for a \$1000 30-year endowment policy issued at age 20.

3. Calculate the first five mean reserves for the policy in problem 1.

4. Calculate the third, tenth, and twenty-third mean reserves for the policy in example 1, section 45.

5. Calculate α and β under the Illinois standard for a \$1000 policy issued at age 35 under each of the following plans:

(*a*) Ordinary life.
(*b*) 20-pay life.
(*c*) 10-pay life.
(*d*) 30-pay life.
(*e*) 25-year endowment.
(*f*) 30-year endowment.
(*g*) 20-year term.

6. Prove analytically that for a given policy the reserve according to the Illinois method is greater than the reserve according to the FPT method at any duration less than n.

46 • THE NEW JERSEY METHOD

A major inconsistency of the Illinois standard is that, for some endowment policies calling for more than 20 premiums, the Illinois standard requires FNL reserves after 20 years at some ages but FPT at others. Consider as an illustration the accompanying table based on CSO 2½%. For ages 27 and under, the 20-pay life premium is

	Net Level Annual Premium for \$1000	
Age at Issue	*20-Pay Life*	*30-Year Endowment*
25	\$24.22551	\$25.09222
26	24.75814	25.26264
27	25.30577	25.44836
28	25.86902	25.65072
29	26.44802	25.87074
30	27.04360	26.11009

less than the 30-year endowment premium; the reverse is true for ages 28 and over. Thus the Illinois standard requires use of the Illinois method for ages 27 and under but requires FPT for ages 28 and over. The accompanying table shows the 30-year endowment reserves for

Age at Issue	*Twentieth Reserve \$1000 30-Year Endowment Illinois Standard*
25	\$575.56
26	575.51
27	575.47
28	565.38
29	565.30
30	565.22

ages 25 to 30 under the Illinois standard, based on CSO $2\frac{1}{2}\%$. The inconsistency is brought out by the sharp break in the twentieth-year reserves between ages 27 and 28.

The New Jersey standard is designed to eliminate this sharp break.

Under the New Jersey standard, policies are divided into three groups:

(*a*) Policies with net level premiums greater than the corresponding net level 20-pay life premium at the same age at issue.

(*b*) Policies with net level premiums less than the corresponding net level 20-pay life premium at the same age at issue but for which the *gross* premium is greater than $1.5c_x$.

(*c*) Other policies.

The standard provides that the Illinois method will be used for group 1, the New Jersey method for group 2, and the FPT method for group 3.

Under the New Jersey method, the first-year reserve is the FPT reserve (i.e., zero). After the first year the reserve is the FPT reserve plus an amount equal to the accumulation of a net level premium, beginning with the second policy year, sufficient to provide for a pure endowment at the end of the twentieth year equal to the difference at that time between the FPT reserve and the FNL reserve. In symbols,

$$_tV^J = {}_tV^F + \pi_J \cdot {}_{t-1}u_{x+1},$$

$$\pi_J \cdot {}_{19}u_{x+1} = {}_{20}V^L - {}_{20}V^F.$$

Alternatively,

$$\alpha_J = c_x, \qquad \beta_J = \beta_F + \pi_J. \tag{69}$$

Equation 58, as applied to the New Jersey method, becomes

$$\alpha_J + \beta_J a_{x:\overline{19}|} = P\ddot{a}_{x:\overline{20}|},$$

$$c_x + \beta_J a_{x:\overline{19}|} = P(1 + a_{x:\overline{19}|}),$$

$$\beta_J a_{x:\overline{19}|} = Pa_{x:\overline{19}|} + (P - c_x),$$

or

$$\beta_J = P + \frac{P - c_x}{a_{x:\overline{19}|}}. \tag{70}$$

Example

Calculate the first five terminal reserves under the New Jersey standard for a \$1000 30-year endowment policy issued at age 30. The gross premium is \$32.70.

Solution: From the first table in this section,

$$P = 26.1101 \text{ and } 1000 \cdot {}_{20}P_{30} = 27.0436.$$

Also,

$$1000c_{30} = 3.4736.$$

Since $P < 1000 \cdot {}_{20}P_{30}$ and the gross premium is greater than $1\frac{1}{2}$ times the natural premium, the policy is in group 2 and the New Jersey method applies.

From equation 69,

$$\alpha_J = 3.4736.$$

From equation 70,

$$\begin{aligned}\beta_J &= 26.1101 + \frac{26.1101 - 3.4736}{a_{30:\overline{19}|}}\\ &= 26.1101 + 1.5828\\ &= 27.6929.\end{aligned}$$

Applying Fackler's method,

$$\begin{aligned}{}_1V^J &= 0.\\ {}_2V^J &= \beta_J u_{31} - 1000k_{31}\\ &= 24.7470.\\ {}_3V^J &= ({}_2V^J + \beta_J)u_{32} - 1000k_{32}\\ &= 50.0271.\\ {}_4V^J &= ({}_3V^J + \beta_J)u_{33} - 1000k_{33}\\ &= 75.8554.\\ {}_5V^J &= ({}_4V^J + \beta_J)u_{34} - 1000k_{34}\\ &= 102.2315.\end{aligned}$$

Check:

$$\begin{aligned}{}_5V^J &= 1000A_{35:\overline{25}|} - \beta_J\ddot{a}_{35:\overline{15}|} - P(\ddot{a}_{35:\overline{25}|} - \ddot{a}_{35:\overline{15}|})\\ &= 1000A_{35:\overline{25}|} - (\beta_J - P)\ddot{a}_{35:\overline{15}|} - P\ddot{a}_{35:\overline{25}|}\\ &= 102.2317.\end{aligned}$$

PROBLEM SET 34

1. Calculate the first five terminal reserves under the New Jersey standard for a $1000 30-year endowment issued at age 35, assuming that the gross premium is greater than the net.

2. Compute the fifth, tenth, and twenty-fifth mean reserves for the policy of problem 1.

3. Compute the fifth, tenth, and twenty-fifth mean reserves for the policy in the example in section 46.

4. Calculate α and β under the New Jersey standard for a $1000 policy issued at age 35 under each of the following plans, assuming that the gross premium is 25% more than the net in all cases.

(*a*) Ordinary life.
(*b*) 20-pay life.
(*c*) 25-year endowment.
(*d*) Endowment at age 70.
(*e*) 15-pay life.
(*f*) Term to age 65.
(*g*) 5-year term.

47 • THE SELECT AND ULTIMATE METHOD

All the valuation methods discussed above make direct provision for heavy first-year expense by defining a first-year valuation net premium which is less than the net level premium. Further, these methods assume that the rate of mortality experienced by the company will follow the tabular rate; that is, actual mortality will agree with the mortality shown by the table used for valuation. The Select and Ultimate method, however, uses an entirely different approach in its effort to achieve the same end result; namely, providing for heavy first-year expenses.

The Select and Ultimate method recognizes that new policyholders are select lives so that the mortality in the early years will be less than the tabular rate. The method contemplates using this mortality saving as an offset against heavy initial expenses.

Under the Select and Ultimate method, the first step is the preparation of a select mortality table under which the mortality in the first policy year is 50% of tabular, 65% in the second policy year, 75% in the third policy year, 85% in the fourth policy year, 95% in the fifth policy year, and 100% after 5 years. These percentages are specified by law and are, of course, arbitrary. The tth terminal reserve is then defined as

$$_tV^S = \text{Value at age } x + t \text{ of future benefits } - P\ddot{a}_{[x]+t:\overline{n-t}|},$$

where P is computed from the original (nonselect) table.

It should be noted that if $t \geqq 5$, $_tV^S = {_tV^L}$.

48 • COMPARISON BETWEEN COMMISSIONERS RESERVES AND ILLINOIS STANDARD RESERVES

The Illinois standard classifies a particular policy according as the net level premium P is greater than the corresponding net level 20-pay life premium or not. The Commissioners method, in its definition of the quantity (a), classifies a particular policy according as the FPT renewal valuation net premium is greater than the corresponding

valuation net premium for a 20-pay life policy. This difference in the two systems complicates our analysis.

It is, unfortunately, not universally true that $P \gtreqless {}_{20}P_x$ according as $\beta_F \gtreqless {}_{19}P_{x+1}$. An algebraic analysis of these relationships leads to no definite conclusion. An empirical analysis indicates that $P - {}_{20}P_x$ usually has the same sign as $\beta_F - {}_{19}P_{x+1}$ but that there are exceptions.

An example, on our usual CSO 2½% assumption, of a policy for which $P > {}_{20}P_x$ but $\beta_F < {}_{19}P_{x+1}$ is a 30-year endowment at age 27. For a \$1000 policy the net level premium is \$25.44836, which is more than the corresponding 20-pay life premium of \$25.30577. On the other hand, the FPT renewal valuation premium is \$26.62404, which is less than the corresponding 20-pay life premium of \$26.85213.

An example of a policy for which $P < {}_{20}P_x$ but $\beta_F > {}_{19}P_{x+1}$ is a 33-pay 48-year endowment issued at age 0. For a \$1000 policy the net level premium is \$16.49227, which is less than the corresponding 20-pay life premium of \$16.49883. But the FPT renewal valuation premium is \$16.22304, which is more than the corresponding 20-pay life premium of \$16.11144. Examples of this type, however, are not easy to find and may for practical purposes be ignored.

If we ignore the cases for which $P < {}_{20}P_x$ but $\beta_F > {}_{19}P_{x+1}$, policies may be divided into four groups:

(*a*) Policies for which $\beta_F \leqq {}_{19}P_{x+1}$ and $P \leqq {}_{20}P_x$.
(*b*) Policies for which $\beta_F \leqq {}_{19}P_{x+1}$ and $P > {}_{20}P_x$.
(*c*) Policies for which $\beta_F > {}_{19}P_{x+1}$ and $n \leqq 20$.
(*d*) Policies for which $\beta_F > {}_{19}P_{x+1}$ and $n > 20$.

GROUP 1. Under the Illinois standard, policies in this group would be valued according to the full preliminary term method.

Under the Commissioners method, following equation 60, we have

$$\beta_C = P + \frac{\beta_F - c_x}{\ddot{a}_{x:\overline{n}|}}.$$

But

$$c_x + \beta_F a_{x:\overline{n-1}|} = P\ddot{a}_{x:\overline{n}|},$$

$$\beta_F \ddot{a}_{x:\overline{n}|} - \beta_F + c_x = P\ddot{a}_{x:\overline{n}|},$$

$$\beta_F = P + \frac{\beta_F - c_x}{\ddot{a}\ _{\overline{n}|}}.$$

Hence, $\beta_C = \beta_F$ under the Commissioners method if $\beta_F \leqq {}_{19}P_{x+1}$. Also $\alpha_C = \alpha_F = c_x$, so that the Commissioners method reserves for this group are full preliminary term.

Thus, for policies in this group, Illinois and Commissioners reserves are equal—assuming the same rates of mortality and interest, of course.

GROUP 2. Under the Illinois standard, policies in this group would be valued according to the Illinois method.

Group 2 is no different from group 1 as far as the Commissioners method is concerned, and policies in this group would therefore be valued by the FPT method.

Since the reserve according to the Illinois method is greater than the reserve according to the FPT method during the premium paying period, we can conclude that the aggregate reserves for policies in group 2 will be greater under the Illinois standard than under the Commissioners method.

GROUP 3. Under the Illinois standard, policies in this group would be valued according to the Illinois method. From equation 68,

$$\beta_I = P + \frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{n}|}}, \text{ since } n \leq 20.$$

From equation 66,

$$\alpha_I = \beta_I - ({}_{19}P_{x+1} - c_x).$$

Under the Commissioners method, we have from equation 60

$$\beta_C = P + \frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{n}|}},$$

and from equation 61,

$$\alpha_C = \beta_C - ({}_{19}P_{x+1} - c_x).$$

Since $\beta_I = \beta_C$ and $\alpha_I = \alpha_C$, the reserves under the two systems are identical for all policies in this group.

GROUP 4. Under the Illinois standard, policies in this group would be valued according to the Illinois method. Thus,

$$\beta_I = P + \frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{20}|}}$$

and

$$\alpha_I = \beta_I - ({}_{19}P_{x+1} - c_x).$$

Under the Commissioners method, we have

$$\beta_C = P + \frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{n}|}}$$

and

$$\alpha_C = \beta_C - ({}_{19}P_{x+1} - c_x).$$

Since $n > 20$, we have

$$\ddot{a}_{x:\overline{n}|} > \ddot{a}_{x:\overline{20}|},$$

$$\frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{n}|}} < \frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{20}|}},$$

$$+ \frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{n}|}} < P + \frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{20}|}},$$

or

$$\beta_C < \beta_I.$$

Also,

$$\beta_C - ({}_{19}P_{x+1} - c_x) < \beta_I - ({}_{19}P_{x+1} - c_x)$$

so that

$$\alpha_C < \alpha_I.$$

Since $\alpha_C < \alpha_I$ and $\beta_C < \beta_I$, it is obvious from a consideration of Fackler's method that Commissioners reserves for policies in this group are less than the corresponding reserves under the Illinois standard. The truth of this statement can also be seen from the fact that under the Illinois method full net level reserves are reached in 20 years whereas under the Commissioners method FNL reserves are not reached until the end of the premium paying period.

CONCLUSION. For policies in groups 1 and 3 the reserves under the two systems are identical. For policies in groups 2 and 4 the reserves are greater under the Illinois standard. Therefore, aggregate reserves on a group of policies are greater under the Illinois standard than under the Commissioners method—assuming use of the same mortality and interest assumptions.

49 • SUPPLEMENTARY READING

The discussion in this chapter has tacitly assumed level premiums and level benefits. For a thorough technical discussion of modified reserve methods, including the problems involved in varying premiums and/or varying benefits, the reader is referred to two papers by Walter O. Menge published in the *Record* of the American Institute of Actuaries. A paper entitled "Preliminary Term Valuation Methods" appears on page 181 of the 1936 *Record* (Volume XXV). A second paper entitled "Commissioners Reserve Valuation Method" appears on page 258 of the 1946 *Record* (Volume XXXV).

For a nontechnical discussion of modified reserve methods the reader is referred to Chapter VII of *Life Insurance* (Sixth Edition) by Joseph B. Maclean, published by McGraw-Hill Book Company, Inc.

PROBLEM SET 35

1. Check the figures in the third and fourth paragraphs of section 48.

2. Prove that aggregate reserves are higher under the New Jersey standard than under the Illinois standard if the mortality and interest assumptions are the same.

3. If $\beta_F < {}_{19}P_{x+1}$, prove that $\beta_C = \beta_F$.

4. If $\beta_C = \beta_F$, prove that $\alpha_C = c_x$.

5. Prove that

$${}_tV^L - {}_tV^C = (P - \beta_C)\,{}_tu_x + (\beta_C - \alpha_C)\,\frac{D_x}{D_{x+t}}.$$

6. Prove that

$${}_tV^L - {}_tV^C = (\beta_C - \alpha_C)\,\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}.$$

7. Prove that ${}_{19:19}V_{x+1} = {}_{20:20}V_x$.

8. If $n \geq 20$, prove that $\pi_I = P - {}_{20}P_x$.

9. If $n \leq 20$, prove that

$$\alpha_I = \alpha_O - \frac{({}_{19}P_{x+1} - P_{x+1})a_{x:\overline{n-1}|}}{\ddot{a}_{x:\overline{n}|}}.$$

10. For a \$1000 policy issued at age 37 the difference between β_O and the net level premium is \$2.04146. Find the length of the premium paying period.

11. Express β_C in terms of α_C, P, x, and n.

12. Calculate α and β for a \$1000 20-year endowment policy issued at age 22 for each of the following valuation standards:

(a) FNL.
(b) FPT.
(c) Ohio.
(d) Canadian.
(e) Illinois.
(f) New Jersey.
(g) CVM.

C H A P T E R   E I G H T

Surrender Values

50 · ASSET SHARE

Let P'_n be the gross premium for a policy for the nth policy year, E_n be the expenses incurred by the policy during the nth policy year, D_n be the dividend paid to the policyholder at the end of the nth policy year, K_n be the cost of insurance based upon the amount at risk for the nth policy year according to the actual mortality experience of the insurance company, and i_n be the rate of interest actually earned by the company during the nth policy year. Assume further, in order to simplify the problem, that P' is actually paid on each policy anniversary, that E_n is incurred at the time the nth premium is paid and includes a reasonable contribution to the surplus or contingency fund of the company, and that K_n is calculated as if all death claims were paid at the end of the policy year of death. Then, if $(AS)_n$ represents the asset share at the end of the nth policy year,

$$
\begin{aligned}
(AS)_1 &= (P'_1 - E_1)(1 + i_1) - K_1 - D_1,\\
(AS)_2 &= [(AS)_1 + P'_2 - E_2](1 + i_2) - K_2 - D_2,\\
(AS)_3 &= [(AS)_2 + P'_3 - E_3](1 + i_3) - K_3 - D_3,\\
&\cdots\\
(AS)_n &= [(AS)_{n-1} + P'_n - E_n](1 + i_n) - K_n - D_n.
\end{aligned}
$$

In words, the asset share may be defined as the accumulated excess of the gross premium over expenses, cost of insurance, and dividends—where the calculations are based on the actual experience of the company rather than on the assumptions made in calculating premiums or reserves.

51 · IDEAL CASH VALUE

It is agreed in theory, and required by law, that a discontinuing policyholder be paid a reasonable amount in cash (or the equivalent in

some other form as discussed in section 53 below). Deciding how much cash is a reasonable amount is not an easy problem. The fundamental problem for the actuary who is constructing a scale of cash surrender values is one of equity. The general principle is that the cash surrender value should be the largest amount that can be paid without leaving the continuing policyholders in a less favorable financial position than if the policy had not been surrendered. This amount is referred to as the "ideal cash value."

The ideal cash value is something less than the asset share. There are two major reasons why the cash value should be something less than the asset share rather than equal to it. First, a certain amount of expense is incurred by the company when the policy is surrendered. This expense should be borne by the discontinuing policyholder, not by the body of continuing policyholders. Second, the company must keep a reasonable proportion of its assets in a liquid condition because of having to pay surrender values in cash and the possibility that many such payments will have to be made in a short period of time. Since a high degree of liquidity in the assets means a lower interest yield, an expense of liquidity is incurred. Again, this expense of liquidity should be borne by the policyholders who actually discontinue their policies.

In actuarial circles the term "rough justice" is sometimes heard. This picturesque term refers to the fact that, as a practical matter, it is impossible to preserve complete equity among the thousands or millions of policyholders in a company, no matter how much the actuary would like to do so. The actuary is forced to do the best he can with the result that perfect equity is only approached. But a reasonable degree of equity—i.e., "rough justice"—is attained. It seems safe to say that complete equity is approached remarkably closely, considering the complexities of the problem and the volume of business.

52 • ADJUSTED PREMIUMS

The Standard Nonforfeiture Law, a part of the previously mentioned Guertin legislation, defines legal minimum cash values in terms of an "adjusted premium," which is similar to the modified net premium used in the Commissioners reserve valuation method. The minimum cash value for a policy at any duration is defined as the present value of future benefits minus the present value of future adjusted premiums. The adjusted premiums are calculated on the assumption that the excess of first-year expense over first-year loading will be amortized out of loadings on renewal premiums over the whole premium paying period, and that this excess is equal to:

(*a*) $20 per $1000 of insurance, plus

(*b*) 40% of the adjusted premium for the first year, but not exceeding $16, plus

(*c*) 25% of the adjusted premium for an ordinary life policy (or for the particular plan, if less) but not exceeding $10.

Then the present value, at date of issue, of all adjusted premiums is equal to the net single premium plus the excess defined above. These figures were designed to produce reasonable cash values for a well-managed company operating on a relatively high expense basis. Companies may pay cash values that are higher than the minimum specified by law.

Example

Calculate the minimum cash value at the end of the fifth year for a $1000 20-pay, 30-year endowment issued at age 30.

Solution: Let P'' be the adjusted premium for an ordinary life policy issued at age 30. Then

$$P''\ddot{a}_{30} = 1000A_{30} + 20 + 0.4P'' + 0.25P'',$$

$$P'' = \frac{1000A_{30} + 20}{\ddot{a}_{30} - 0.65}$$

$$= \frac{433.80049}{23.384180}$$

$$= \$18.5510.$$

Note: If P'' had turned out to be greater than $40, we would have to recalculate it, using $26 instead of $0.65P''$ because of the limitations in (*a*) and (*b*) in the definition of the excess.

Now let P''' be the adjusted premium for the policy in question. Then

$$P'''\ddot{a}_{30:\overline{20|}} = 1000A_{30:\overline{30|}} + 20 + 0.4P''' + 0.25P'',$$

$$P''' = \frac{1000A_{30:\overline{30|}} + 20 + 0.25P''}{\ddot{a}_{30:\overline{20|}} - 0.4}$$

$$= \frac{541.6658}{14.901234}$$

$$= \$36.3504.$$

Again, if P''' had been greater than $40 or less than P'', we would have to recalculate.

The fifth-year minimum cash value

$$= 1000A_{35:\overline{25|}} - P'''\ddot{a}_{35:\overline{15|}}$$

$$= 575.73071 - 443.53282$$

$$= \$132.20.$$

PROBLEM SET 36

Find the minimum cash value at the end of the tenth year for a $1000 policy issued at age 50 under each of the following plans: (1) ordinary life; (2) 20-pay life; (3) 15-year endowment.

53 · REDUCED PAID-UP AND EXTENDED TERM INSURANCE

Most policies provide that the withdrawing policyholder may elect the reduced paid-up option or the extended term option in lieu of cash. If the reduced paid-up option is chosen, the withdrawing policyholder receives paid-up insurance on the same plan as his original policy. For example, if a policyholder should lapse a 20-pay, 30-year endowment issued at age 30 after 5 years and elect the paid-up option, he would be insured for a reduced amount on the endowment at age 60 plan. The amount would be determined from the equation

$$(\text{Cash value}) = (\text{Amount of reduced paid-up insurance}) \cdot A_{35:\overline{25|}}$$

In general, the amount of paid-up insurance is whatever the cash value used as a single premium will purchase at the attained age of the lapsing policyholder. The rates are net; i.e., it is assumed that there are no expenses.

Under the extended term insurance option, the cash value is used to purchase, at net rates, term insurance for the full face amount of the policy. The length of the term depends on the age at withdrawal and the amount of the cash value. In the case of endowment insurance it may happen that the cash value is more than enough to purchase term insurance to the end of the original term. In this case the part of the cash value not needed to purchase term insurance is used to purchase a pure endowment payable at the end of the term. The Guertin laws provide that in calculating extended term benefits mortality rates not exceeding 130% of CSO may be used. In our examples we shall assume 100% of CSO.

Calculating reserves on reduced paid-up and extended term poses no particular problem. Since the benefits are fully paid, the reserve is the net single premium at the attained age for whatever benefit is then provided.

EXAMPLE 1

In section 45 the cash value at the end of the fifth year for a \$1000 20-pay, 30-year endowment policy was calculated. Calculate the other two surrender options.

Solution: Let $x =$ the amount of reduced paid-up insurance. Then

$$132.20 = x \cdot A_{35:\overline{25}|},$$

$$x = \frac{132.20}{0.57573071}$$

$$= \$230.$$

Note: Reduced paid-up is ordinarily calculated to the nearest dollar or the next higher dollar. The former rule will be followed in these illustrations.

To find the term of extended insurance interpolate in a table for $1000A^{\ 1}_{35:\overline{n}|}$. From Table 10, we have

$$1000A^{\ 1}_{35:\overline{20}|} = 124.80,$$

$$1000A^{\ 1}_{35:\overline{25}|} = 172.46.$$

Since the amount we have on hand to purchase term insurance is 132.20, the term is somewhere between 20 and 25 years. For obvious practical reasons the tables in the back of the book are not complete. We must, therefore, calculate $1000A^{\ 1}_{35:\overline{n}|}$ for various values of n between 20 and 25. We can quit as soon as we have 132.20 included between two consecutive values.

$$1000A^{\ 1}_{35:\overline{21}|} = 1000\frac{M_{35} - M_{56}}{D_{35}}$$

$$= 133.70.$$

Luckily $1000A^{\ 1}_{35:\overline{20}|} < 132.20 < 1000A^{\ 1}_{35:\overline{21}|}$, and we can quit calculating values of $1000A^{\ 1}_{35:\overline{n}|}$ already. We have

$$1000A^{\ 1}_{35:\overline{21}|} = 133.70$$

$$1000A^{\ 1}_{35:\overline{20}|} = \underline{124.80}$$

$$\text{Difference} = 8.90.$$

$$\text{Daily difference} = \frac{8.90}{365} = 0.02438.$$

$$\text{Number of days} = \frac{132.20 - 124.80}{0.02438}$$

$$= 304.$$

Therefore the term of the extended insurance is 20 years, 304 days.

EXAMPLE 2

Calculate all three surrender options if a $1000 20-pay, 30-year endowment issued at age 30 is surrendered after 15 years. Assume legal minimum cash values.

Solution: From the example of section 52 the adjusted premium is 36.3504.

The cash value

$$= 1000A_{45:\overline{15|}} - 36.3504\ddot{a}_{45:\overline{5|}}$$
$$= 713.51244 - 170.00566$$
$$= \$543.51.$$

$$\text{Reduced paid-up insurance} = \frac{543.51}{A_{45:\overline{15|}}} = \$762.$$

Since $1000A^{\ 1}_{45:\overline{15|}} = 164.60$, the cash value will provide term insurance for 15 years (i.e., to the end of the term of the original policy) and leave $543.51 - 164.60 = 378.91$ to purchase a 15-year pure endowment amounting to

$$\frac{378.91}{{}_{15}E_{45}} = \$690.$$

Therefore the term of extended insurance is 15 years and the amount of pure endowment, payable at the end of the term, is $690.

PROBLEM SET 37

Use the results of problem set 36 to calculate the reduced paid-up and extended term options at the end of the tenth year for a $1000 policy issued at age 50 under each of the following plans: (1) ordinary life; (2) 20-pay life; (3) 15-year endowment.

PROBLEM SET 38

1. Calculate the minimum cash values and the corresponding insurance options at the end of the fifth year for a $1000 policy issued at age 24 under each of the following plans:

(*a*) Ordinary life.
(*b*) Life paid-up at age 65.
(*c*) Endowment at 85.
(*d*) Endowment at 60.
(*e*) 10-pay life.
(*f*) 20-year endowment.
(*g*) Single-premium life.

2. Prepare a schedule showing FNL and CVM terminal reserves, minimum cash values, and the insurance surrender options for the first 12 durations under the following $1000 policies issued at age 21:

(*a*) 5-year endowment.
(*b*) 10-year endowment.
(*c*) 5-pay life.
(*d*) 10-pay life.

C H A P T E R   N I N E

Gross Premiums

54 · NONPARTICIPATING AND PARTICIPATING POLICIES

Two kinds of policies are issued by life insurance companies, nonparticipating and participating. The premium rates and benefits for a nonparticipating policy are guaranteed; the policy does not participate in the earnings of the company. Rates and benefits for a participating policy are also guaranteed but in a somewhat different sense. The guarantees are in the nature of minimum guarantees. It is expected when the policy is issued that the effective premium will be reduced by dividends. Sometimes benefits are increased beyond those guaranteed in the policy. For example, some companies grant a surrender dividend to a withdrawing policyholder so that the actual surrender value is greater than that guaranteed.

Since participating policies contemplate that the effective premium will be reduced by dividends, and, since nonparticipating policies contemplate no dividends, participating gross premiums are invariably higher than nonparticipating gross premiums if the benefits are the same.

55 · PARTICIPATING GROSS PREMIUMS

In determining a scale of participating gross premiums the primary consideration is that the premiums be easily sufficient to provide the benefits and pay expenses. It is not necessary that a premium be calculated with extreme refinement because the part of the premium not needed for the benefits and expenses will be refunded in the form of a dividend. The usual procedure is to set the gross premium equal to the net premium plus the loading where the net premium is based on the same interest and mortality assumptions used in figuring reserves and where the loading is computed by a relatively simple formula. As an example, the gross premium for a 20-pay life policy issued at

age 30 might be set equal to

$$1.125\,{}_{20}P_{30} + 0.125P_{30}$$

where the net premiums are based on CSO 2½%.

The reader should note that the word "dividend" has a different meaning when applied to a life insurance policy than when applied to a share of stock. A dividend paid to the holder of a share of stock is a return on his investment; a dividend paid to a life insurance policyholder is a refund of part of the premium—the part not needed for benefits and expenses according to the actual experience of the insurance company.

Since participating insurance is on a sound basis if gross premiums are sufficient, almost any intelligent person with or without actuarial training could set a scale of premiums which would work, provided the dividend scale is calculated on sound actuarial principles. However, current actuarial thought is in the direction of making participating gross premiums as equitable as possible so that less burden is placed on the dividend scale. The principles involved in setting a scale of participating gross premiums are thoroughly discussed in a paper by H. R. Bassford entitled "Premium Rates, Reserves, and Non-Forfeiture Values for Participating Policies." The paper begins on page 328 of the 1942 *Transactions* (Vol. XLIII) of the Actuarial Society of America.

56 • NONPARTICIPATING GROSS PREMIUMS

The problem of calculating nonparticipating gross premiums is entirely different from that of participating premiums and is generally conceded to be more difficult. Since the problem of equity as among different types of policies cannot be placed on the shoulders of the dividend scale, the gross premiums must be as equitable as possible. Further, the actuary who sets the premium scale must steer a careful course between two objectives. The premiums must be low enough to be competitive but high enough to be safe and to yield a reasonable profit. Therefore, nonparticipating premiums are based on a table of mortality, a rate of interest, and a set of expense factors which, in the judgment of the actuary, are good estimates of what the company may expect in the future. Making such estimates is the first step in building a set of premium rates.

There are three principal methods of computing nonparticipating gross premiums. The essential difference depends largely on the treatment of the profit which a company may legitimately expect from its nonparticipating business. Under the first method the gross

premium is calculated on the most probable assumptions and a specific amount added for expected profit. The second method uses factors that are more conservative than the most probable but not so conservative as would be used for participating premiums. The expected profit under this second method arises from the difference between the assumed factors and the most probable rather than from a specific part of the loading. Under the third method the premium is calculated upon the assumption that the most probable mortality, interest, and expense factors will hold for a certain number of years, say n, and that at the end of n years the policy will mature as an endowment for the nth year reserve. The profit in this third method arises from the excess of the nth year legal reserve over the amount actually needed at the end of n years in order to meet the contract obligations. (The reader should understand that in nonparticipating insurance the mortality and interest assumptions used in calculating the legal reserve are not the same as those used in calculating the premium.) Our discussion will be confined to the first of these three methods.

57 • MORTALITY

Persons who qualify for life insurance represent a body of select lives. Insurance company studies show conclusively that the mortality is lower for persons aged x to whom policies have recently been issued than for persons aged x to whom policies have been issued some time ago. Since nonparticipating premiums are calculated on the most probable assumptions, a select table is clearly in order. Although companies do not all use the same table, enough of them currently use Table Z with a select period grafted on so that this table may be considered representative of current (early 1949) practice. When Table Z is used, its use is usually confined to life and endowment plans. The reason is that mortality under term plans is greater than under life and endowment plans—presumably because of selection against the company—so that a table showing higher rates of mortality must be used for term plans.

There is one other point in connection with deaths which must be mentioned. Net premiums—i.e., premiums that assume that expenses are zero—are usually calculated on the assumption that claims are paid at the end of the policy year of death. This assumption is not in accordance with the facts. Claims are actually paid as soon as proper proof of death has been filed with the company. Although there is a little lag between date of death and date of payment, the usual assumption is that deaths occur on the average in the middle of the policy year and that death claims are paid immediately on death. This

means that on the average claims are paid 6 months before the end of the policy year so that a 6-month loss of interest, not contemplated in the net premium calculation, is suffered by the company. Thus, paying a \$1000 death claim at date of death is equivalent to paying $1000(1 + i)^{1/2}$ at the end of the year, where i is the rate of interest used in calculating the net premium. $\left(1 + \frac{i}{2}\right)$ is frequently used in practice instead of $(1 + i)^{1/2}$.

58 · INTEREST

In recent years the rate of interest to be used in the premium calculations has presented a very difficult problem. Beginning about 1930 and continuing to about 1948 the trend of the interest rate was steadily downward. Since 1948 the rate has shown some signs of turning upward, but few persons in 1949 would be willing to stake their lives on predicting within narrow limits where the interest rate will be in 1955, for example. Companies, of course, differ with respect to both the rate of return currently being earned and the estimate for the future. However, 2¾% to 3% may be taken as representative of the interest assumptions currently used in nonparticipating premium calculations. Some actuaries prefer to assume one rate for the first few years in the future and a more conservative rate for the more distant years.

59 · EXPENSES

The major expenses involved in connection with a life insurance policy are:

(*a*) Commissions to the soliciting agent.
(*b*) Premium tax.
(*c*) Federal income tax.
(*d*) Fee for medical examination and inspection report.
(*e*) Home and branch office clerical expense and overhead.

Some of these expenses are relatively easy to allocate; others are very difficult.

Although a company may use a combination salary and commission arrangement for new agents, most agents are compensated on a straight commission basis. Since the commission scale is definite and is expressed as a percentage of premium, there is no particular difficulty in predicting what the commission for a particular policy is going to be even though the commission percentage varies by plan and duration.

The premium tax paid to the states is figured as a percentage of

premium (with or without dividend credit) in almost all states. The tax rate may vary from year to year according to the whims of the various state legislatures, but the average rate can be expected to remain fairly constant. Premium tax and commissions are called percentage expenses since they are expressed as a percentage of the premium. The percentage expenses for a $10,000 policy are exactly 10 times (in dollars) the percentage expenses for a similar $1000 policy.

The expense items *d* and *e* above are called constant expenses because they do not vary much by premium. These expenses would be just about the same for a high premium policy, such as 20-year endowment, as for ordinary life. These expenses do, however, vary by size of policy. It costs more, on the average, to handle a $25,000 policy than a $1000 policy because a more thorough medical examination may be required for the $25,000 policy, a complicated settlement option to take effect at death may be required by the larger policy, etc. But it does not cost 25 times as much. It could be argued that, the larger the policy, the lower the expense factor per $1000 of insurance. In practice, however, the expense factors for a particular plan and age are based on the average-sized policy at that plan and age, and the expense charge for a $25,000 policy is 25 times as much as the expense charge for a $1000 policy. Three reasons for not distinguishing policies by size when it comes to allocating expenses may be cited:

(*a*) Legal restrictions.
(*b*) Practical administrative considerations.
(*c*) Statistics which indicate that mortality runs higher on the larger policies. To a certain extent the lower expenses and higher mortality offset each other.

Deciding on the actual constant expense factors is a difficult problem. Inflation or deflation can increase or decrease these expenses, whereas the premium is fixed by contract. Moreover, the life insurance business does not lend itself readily to accurate cost accounting. The judgment of the actuary must necessarily play an important part in setting the expense factors.

The Federal income tax raises an interesting question. Federal income tax is based on investment income. This gives rise to two points of view. One is that the income tax is really just a license for doing business, that the method of calculation is beside the point, and that the expense should be treated as a constant expense. Thus a low premium policy would contribute just as much toward Federal income tax as would a high premium policy. The second point of

view is that, since the tax is based on investment income, it should be treated as an offset to interest and not considered in the expenses at all. The first point of view probably makes the most sense in theory; the second perhaps leads to the most practical solution. In the illustrations in this chapter the second approach will be adopted.

60 • THE CAMMACK-TYPE FORMULA

A paper by E. E. Cammack entitled "Premiums for Non-Participating Life Insurances" appeared on page 379 of the 1919 *Transactions* (Vol. XX) of the Actuarial Society of America. Despite the fact that this paper is 30 years old, it is still the leading reference on the subject and the principles contained in the paper are still valid today. Formulas based on these principles are referred to as Cammack-type formulas.

Essentially, the Cammack-type formula says that the present value of future gross premiums is equal to the present value of future benefits plus the present value of future expenses, where all present values are calculated as of the date of issue. The examples below show how the formula works. These examples, and the subsequent problems on nonparticipating premiums, assume that mortality follows a select table in which the effect of selection is assumed to wear off after 5 years. The expenses assumed are purely illustrative and do not represent the expenses of any actual company. Interest at 3% is assumed.

In the Cammack-type formula withdrawal rates are ignored on the theory that the cash surrender value will be so close to the ideal surrender value that the overall gain or loss from surrenders is insignificant. If a company's cash surrender values are such that this assumption does not fit the facts, it is necessary to take surrender values into account in calculating gross premiums. Actuarial literature contains several papers covering the subject of gross premiums taking surrender values into account.

Example 1

Under a $1000 ordinary life policy issued at age 30 the following expenses are estimated:

(*a*) Commissions: 40% first year, 20% second year, 10% third year, 2% thereafter.

(*b*) Premium tax: 2%.

(*c*) Administrative and underwriting expenses: $5.00 first year, $2.00 thereafter.

It is further assumed that all expenses are incurred at the start of each policy year. Express the annual gross premium in select commutation symbols.

Note: The only difference between select notation and ultimate notation lies in the way the age subscripts are written. Thus, in select notation

$q_{[x]}$ represents the probability that a person aged x who has just qualified for a life insurance policy will die before attaining age $x + 1$;

$q_{[x]+n}$ represents the probability that a person now aged $x + n$ who qualified for life insurance n years ago will die before attaining age $x + n + 1$;

$D_{[x]+n}$ represents the D function for a group of lives now aged $x + n$ who qualified for life insurance n years ago;

and so on.

If n is greater than the select period, which we assume to be 5 years in our examples, the brackets are omitted. Thus,

$q_{[32]}$ represents the probability that a person aged 32 who has just qualified for life insurance will die before attaining age 33;

$q_{[30]+2}$ represents the probability that a person aged 32 who qualified for life insurance 2 years ago will die before attaining age 33;

q_{32} represents the probability that a person aged 32 who qualified for life insurance at least 5 years ago will die before attaining age 33;

and so on.

Solution: Let P' represent the gross premium. Then

$$P'\ddot{a}_{[30]} = 1015A_{[30]} + 0.4P' + 0.2P' \cdot {}_1E_{[30]} + 0.1P' \cdot {}_2E_{[30]} + 0.02P' \cdot {}_{3|}\ddot{a}_{[30]} + 0.02P'\ddot{a}_{[30]} + 3 + 2\ddot{a}_{[30]}.$$

Changing the functions to commutation symbols, and multiplying through by $D_{[30]}$, we have

$$P'N_{[30]} = 1015M_{[30]} + 0.4P'D_{[30]} + 0.2P'D_{[30]+1} + 0.1P'D_{[30]+2} + 0.02P'N_{[30]+3} + 0.02P'N_{[30]} + 3D_{[30]} + 2N_{[30]},$$

$$P'(N_{[30]} - 0.4D_{[30]} - 0.2D_{[30]+1} - 0.1D_{[30]+2} - 0.02N_{[30]+3} - 0.02N_{[30]}) = 1015M_{[30]} + 3D_{[30]} + 2N_{[30]},$$

$$P' = \frac{1015M_{[30]} + 3D_{[30]} + 2N_{[30]}}{0.98N_{[30]} - 0.4D_{[30]} - 0.2D_{[30]+1} - 0.1D_{[30]+2} - 0.02N_{[30]+3}}.$$

In order to produce a numerical answer all we have to do is have a table of select commutation values on the assumed mortality basis and substitute the numbers in the above expression.

EXAMPLE 2

Under a \$1000 10-pay, 20-year endowment policy issued at age 30 the following expenses are estimated:

(a) Commissions: 25% first year, 10% second year, 5% third year, 2% thereafter.
(b) Premium tax: 2%.
(c) Administrative and underwriting expenses: \$5.00 first year, \$2.00 thereafter during the premium paying period, \$1.00 after the policy is paid-up.

Express the annual gross premium in select commutation symbols.

Solution:

$$
\begin{aligned}
P'\ddot{a}_{[30]:\overline{10}|} = {} & 1015A^{\;1}_{[30]:\overline{20}|} + 1000 \cdot {}_{20}E_{[30]} + 0.25P' + 0.1P' \cdot {}_{1}E_{[30]} \\
& + 0.05P' \cdot {}_{2}E_{[30]} + 0.02P'\,{}_{3|7}\ddot{a}_{[30]} + 0.02P'\ddot{a}_{[30]:\overline{10}|} \\
& + 3 + \ddot{a}_{[30]:\overline{10}|} + \ddot{a}_{[30]:\overline{20}|},
\end{aligned}
$$

$$
\begin{aligned}
P'(N_{[30]} - N_{40}) = {} & 1015(M_{[30]} - M_{50}) + 1000D_{50} + 0.25P'D_{[30]} \\
& + 0.1P'D_{[30]+1} + 0.05P'D_{[30]+2} + 0.02P'(N_{[30]+3} \\
& - N_{40}) + 0.02P'(N_{[30]} - N_{40}) + 3D_{[30]} + (N_{[30]} \\
& - N_{40}) + (N_{[30]} - N_{50}),
\end{aligned}
$$

$$
\begin{aligned}
& P'(N_{[30]} - N_{40} - 0.25D_{[30]} - 0.1D_{[30]+1} - 0.05D_{[30]+2} - 0.02N_{[30]+3} \\
& + 0.02N_{40} - 0.02N_{[30]} + 0.02N_{40}) = 1015(M_{[30]} - M_{50}) + 1000D_{50} \\
& \qquad + 3D_{[30]} + N_{[30]} - N_{40} + N_{[30]} - N_{50},
\end{aligned}
$$

$$
P' = \frac{1015(M_{[30]} - M_{50}) + 1000D_{50} + 3D_{[30]} + 2N_{[30]} - N_{40} - N_{50}}{0.98N_{[30]} - 0.96N_{40} - 0.02N_{[30]+3} - 0.25D_{[30]} - 0.1D_{[30]+1} - 0.05D_{[30]+2}}.
$$

PROBLEM SET 39

1. Under a \$1000 20-pay life policy issued at age 30 the following expenses are estimated:

(a) Commissions: 35% first year, 15% second year, 10% third year, 2% thereafter.
(b) Premium tax: 2%
(c) Administrative and underwriting expenses: \$5.00 first year, \$2.00 thereafter during the premium paying period, \$1.00 after the policy is paid up.

Express the annual gross premium in select commutation symbols.

2. Under a \$1000 40-year endowment policy issued at age 30 the following expenses are estimated:

(a) Commissions: 50 % first year, 5 % for the next 9 years, nothing thereafter.
(b) Premium tax: 2 %
(c) Administrative and underwriting expenses: \$6.00 first year, \$2.10 thereafter.

Express the annual gross premium in select commutation symbols.

3. Under a \$1000 ordinary life policy issued at age 20 the following expenses are estimated:

(a) Commissions: 55 % first year, 5 % for the next 4 years, 2 % thereafter.
(b) Premium tax: 2 %.
(c) Administrative and underwriting expenses: \$6.00 first year, \$2.00 thereafter.

Express the annual gross premium in select commutation symbols.

4. Under a \$1000 10-pay life policy issued at age 10 the following expenses are estimated:

(a) Commissions: 25 % first year, 5 % thereafter.
(b) Premium tax: 2 %.
(c) Administrative and underwriting expenses: \$7.00 first year, \$3.00 thereafter during the premium paying period, \$1.00 after the policy is paid up.

Express the annual gross premium in select commutation symbols.

5. Under a single-premium life policy issued at age 40 the following expenses are estimated:

(a) Commissions: 3 %.
(b) Premium tax: 2 %.
(c) Administrative and underwriting expenses: \$8.00 first year, \$1.00 thereafter.

Express the gross single premium in select commutation symbols.

6. The loading formula for a participating 20-pay life policy is described as 10 % of the gross premium plus 15 % of the net premium for an ordinary life policy issued at the same age. Calculate the gross premium for a \$1000 policy issued at age 30.

7. The loading formula for a 30-year participating endowment policy is described as \$5 plus 2 % of the age at issue per thousand plus 10 % of the gross premium. Calculate the gross premium for a \$1000 policy issued at age 25.

8. Express in commutation symbols the net single premium for whole-life insurance if 3 % interest is assumed for the first 10 years and 2½ % thereafter.

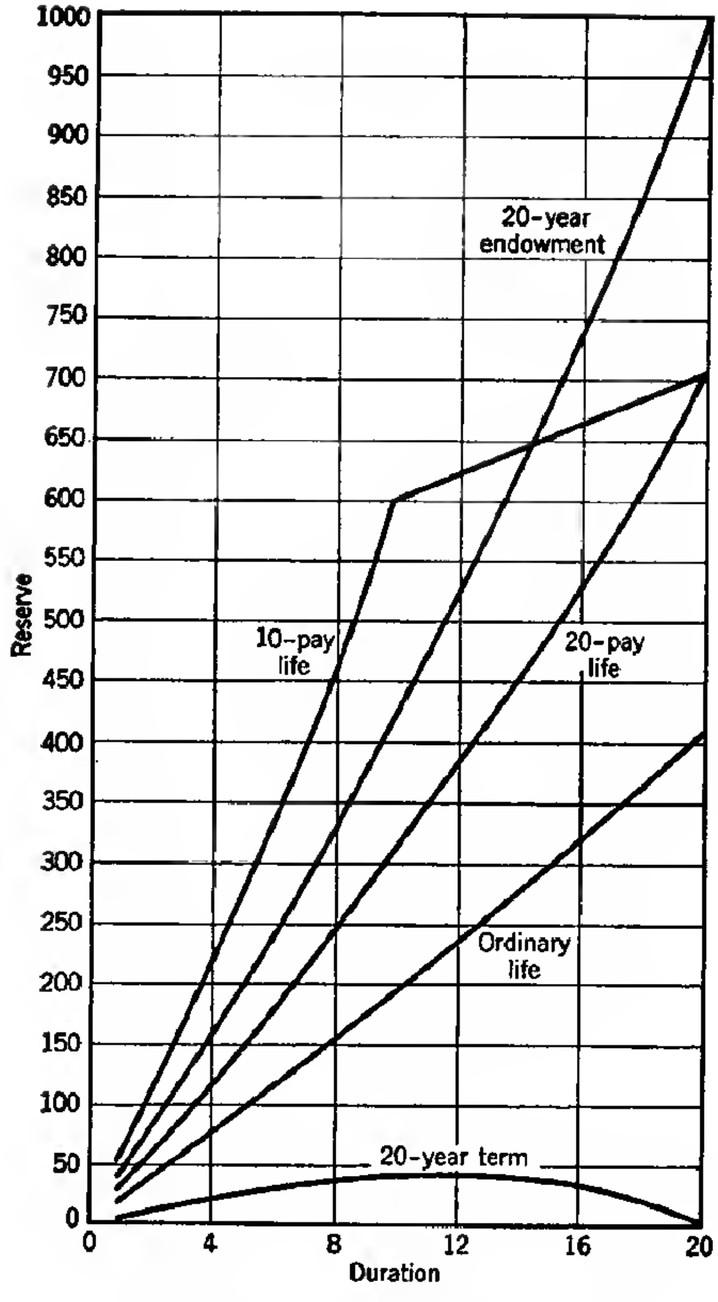
APPENDIX ONE

Net Level Terminal Reserves

It is interesting to compare the progress of the reserves for various plans of insurance. The table below shows the net level terminal reserves for five common plans of insurance, all issued at age 40 with a face amount of $1000. In order to facilitate visual comparison, this table has been graphed. The graph appears on the next page.

NET LEVEL TERMINAL RESERVES—CSO 2½%
AGE 40—FACE AMOUNT $1000

Duration	*Ordinary Life*	*20-Pay Life*	*10-Pay Life*	*20-Year Endowment*	*20-Year Term*
1	$ 19.20	$ 29.00	$ 53.43	$ 38.69	$ 5.88
2	38.61	58.52	108.17	78.21	11.54
3	58.22	88.57	164.29	118.60	16.94
4	78.02	119.17	221.83	159.88	22.05
5	97.98	150.31	280.88	202.09	26.80
6	118.11	182.03	341.51	245.27	31.15
7	138.37	214.32	403.83	289.48	35.05
8	158.76	247.22	467.94	334.75	38.42
9	179.26	280.74	533.96	381.16	41.21
10	199.85	314.92	602.03	428.78	43.31
11	220.51	349.78	612.31	477.69	44.65
12	241.21	385.36	622.61	528.00	45.12
13	261.96	421.73	632.93	579.82	44.61
14	282.70	458.92	643.24	633.29	42.98
15	303.44	497.02	653.56	688.56	40.10
16	324.14	536.10	663.85	745.83	35.79
17	344.77	576.27	674.12	805.33	29.86
18	365.33	617.64	684.34	867.30	22.09
19	385.78	660.37	694.51	932.07	12.24
20	406.10	704.62	704.62	1000.00	0



Net Level Terminal Reserves—CSO 2½ %
Age 40—Face Amount $1000

APPENDIX TWO

Comparison of Natural and Net Level Premiums

The graph on the next page shows an interesting relation between the net level premium (ordinary life) and the natural premiums. The graph was constructed by plotting a point to represent the net premium for 1-year term insurance for every age from 50 to 99, inclusive, and plotting another point to represent the net premium for ordinary life insurance (issued at age 50) for the same ages. The fifty points representing the natural premiums at the various ages were connected by a smooth curve, as were the fifty points representing the net level premium. In the latter case the curve is, of course, a straight line parallel to the base of the graph, since the ordinary life premium is $36.90 at each attained age.

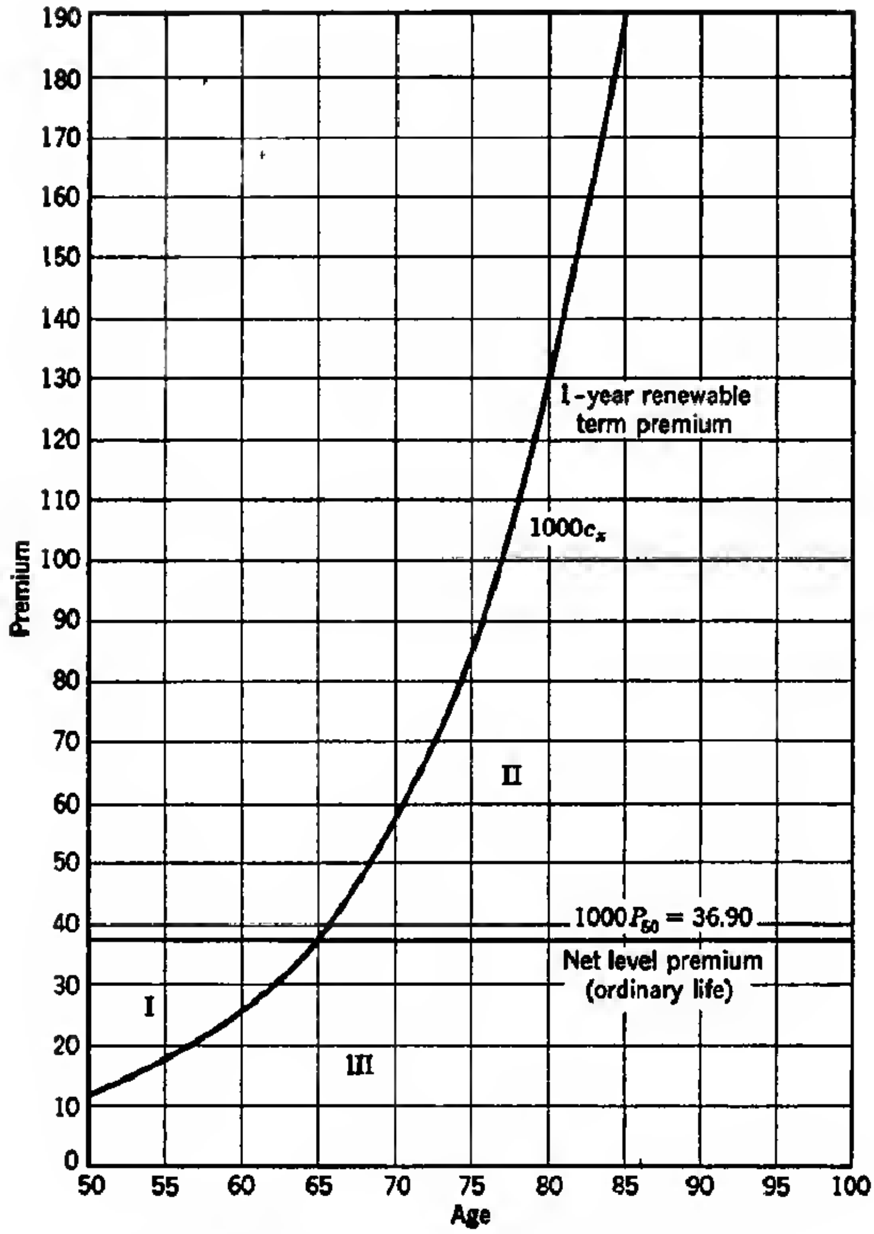
In order to keep the scale from being too small the highest age for which the natural premium is shown is 85. The premium continues to increase after that age, reaching $975.61 at age 99.

The only reason for drawing a curve through the isolated points is to aid the reader in making a visual comparison. It should not be inferred that the functions are continuous.

An illustration of this type is frequently used in explaining (1) the necessity for building up reserves on a level premium policy, and (2) the origin of the reserves. The usual explanation runs as follows:

Since the two curves cross between ages 64 and 65 (in this particular case), it appears that the net level premium is more than enough to pay death claims until age 65, after which the net level premium is not enough to pay death claims. The company must, therefore, withhold the "excess" from the early years in order to make up the "deficit" in the later years.

This explanation sometimes gives rise to the question of why the "excess" in the early years, as represented on the graph by the area I bounded by the two curves and the left-hand border, is obviously less than the "deficit" in the later years, as represented by the area II bounded by the two curves and the right-hand border. This question can be answered by an analysis of the graph.



Graphic Comparison of Net Premiums—CSO 2½ %
Age 50—Face Amount $1000

Since area II is apparently greater than area I, area II + area III > area I + area III. But area II + area III is the area under the $1000c_x$ curve. If we remember that the curves are only visual aids drawn through isolated points, we see that the area under the $1000c_x$ curve may be represented by the sum of the natural premiums; i.e., $1000c_{50} + 1000c_{51} + 1000c_{52} + \cdots + 1000c_{99}$. Similarly, the area under the $1000P_{50}$ curve may be represented by $1000P_{50} + 1000P_{50} + 1000P_{50} + \cdots + 1000P_{50} = 50(1000P_{50})$. Thus, observing that area II > area I amounts to observing that the sum of the natural premiums is greater than the sum of the net level premiums—or, in symbols, that $c_{50} + c_{51} + c_{52} + \cdots + c_{99} > 50P_{50}$. The explanation reduces, therefore, to explaining this inequality.

The benefits under an ordinary life policy and a 1-year renewable term arrangement for the same face amount are identical. Both provide for a payment of the face amount at the end of the year of death. Since the benefits are identical, the present value of the premiums must be identical. Thus,

$$\begin{aligned} &c_{50} + {}_1E_{50} \cdot c_{51} + {}_2E_{50} \cdot c_{52} + \cdots + {}_{14}E_{50} \cdot c_{64} \\ &\quad + {}_{15}E_{50} \cdot c_{55} + \cdots + {}_{49}E_{50} \cdot c_{99} = P_{50} + {}_1E_{50} \cdot P_{50} \\ &+ {}_2E_{50} \cdot P_{50} + \cdots + {}_{14}E_{50}P_{50} + {}_{15}E_{50} \cdot P_{50} + \cdots + {}_{49}E_{50} \cdot P_{50}. \end{aligned}$$

or

$$\begin{aligned} &(P_{50} - c_{50}) + {}_1E_{50}(P_{50} - c_{51}) + {}_2E_{50}(P_{50} - c_{52}) + \cdots \\ &\qquad + {}_{14}E_{50}(P_{50} - c_{64}) = {}_{15}E_{50}(c_{65} - P_{50}) + {}_{16}E_{50}(c_{66} - P_{50}) \\ &\qquad\qquad + {}_{17}E_{50}(c_{67} - P_{50}) + \cdots + {}_{49}E_{50}(c_{99} - P_{50}), \end{aligned}$$

which is to say that the present values of the "excesses" equal the present values of the "deficits." Since $1 > {}_1E_{50} > {}_2E_{50} > \cdots > {}_{49}E_{50}$, the discount factors for the "excesses" are considerably greater than the discount factors for the "deficits." In order for the equality to hold, therefore, the sum of the "deficits" must be greater than the sum of the "excesses." That is,

$$\begin{aligned} &(c_{65} - P_{50}) + (c_{66} - P_{50}) + (c_{57} - P_{50}) + \cdots + (c_{99} - P_{50}) \\ &\qquad > (P_{50} - c_{50}) + (P_{50} - c_{51}) + (P_{50} - c_{52}) + \cdots \\ &\qquad\qquad + (P_{50} - c_{54}), \end{aligned}$$

or

$$c_{50} + c_{51} + c_{52} + \cdots + c_{99} > 50P_{50}.$$

It is possible to give a formal mathematical proof of the inequality

$$c_x + c_{x+1} + c_{x+2} + \cdots + c_\omega > (\omega - x + 1)P_x$$

provided that

$$q_x < q_{x+1} < q_{x+2} < \cdots < q_\omega.$$

This continued inequality (q increasing with age) is true for mortality tables generally, except at very young ages and for the CSO table in particular for ages 11 and over. The proof employs the method of mathematical induction and is given below.

THEOREM

$$c_x + c_{x+1} + c_{x+2} + \cdots + c_\omega > (\omega - x + 1)P_x, \quad \text{if } q_x < q_{x+1} < q_{x+2} < \cdots < q_\omega.$$

We shall prove the theorem by proving the following lemma, of which the theorem is really a special case ($n = \omega - x + 1$).

LEMMA

$$\frac{C_x}{D_x} + \frac{C_{x+1}}{D_{x+1}} + \frac{C_{x+2}}{D_{x+2}} + \cdots + \frac{C_{x+n-1}}{D_{x+n-1}} > n\,\frac{C_x + C_{x+1} + C_{x+2} + \cdots + C_{x+n-1}}{D_x + D_{x+1} + D_{x+2} + \cdots + D_{x+n-1}},$$

if

$$q_x < q_{x+1} < q_{x+2} < \cdots < q_{x+n-1}.$$

We shall prove the lemma by assuming that it holds for $n = k$, proving that it holds for $n = k + 1$, as a logical consequence of its holding for $n = k$, and complete the induction by proving that the lemma holds for $n = 2$.

Assume that the lemma holds for $n = k$. Then

$$\sum_{i=0}^{k-1} \frac{C_{x+i}}{D_{x+i}} > k\,\frac{\sum_{i=0}^{k-1} C_{x+i}}{\sum_{i=0}^{k-1} D_{x+i}} = k\,\frac{M_x - M_{x+k}}{N_x - N_{x+k}} = k\,\frac{A}{B},$$

where $A = M_x - M_{x+k}$ and $B = N_x - N_{x+k}$. The purpose of this substitution is to cut down the algebraic detail.

The next step in the induction is to show that, if the theorem holds for $n = k$, it holds for $n = k + 1$. In other words, we must prove that, if $\sum_{i=0}^{k-1} \frac{C_{x+i}}{D_{x+i}} > k\frac{A}{B}$, then $\sum_{i=0}^{k} \frac{C_{x+i}}{D_{x+i}} > (k+1)\frac{A + C_{x+k}}{B + D_{x+k}}$. This amounts to showing that $\sum_{i=0}^{k} \frac{C_{x+i}}{D_{x+i}} - (k+1)\frac{A + C_{x+k}}{B + D_{x+k}}$ is positive.

In order to simplify the rather involved algebra as much as possible, we shall let $C = C_{x+k}$ and $D = D_{x+k}$.

Now,

$$\sum_{i=0}^{k} \frac{C_{x+i}}{D_{x+i}} - (k+1)\frac{A + C_{x+k}}{B + D_{x+k}}$$

$$= \sum_{i=0}^{k-1} \frac{C_{x+i}}{D_{x+i}} + \frac{C}{D} - k\frac{A+C}{B+D} - \frac{A+C}{B+D}$$

$$= \sum_{i=0}^{k-1} c_{x+i} + \frac{C}{D} - \frac{A+C}{B+D} - k\frac{AB + BC + AD - AD}{B(B+D)}$$

$$= \sum_{i=0}^{k-1} c_{x+i} + \frac{C}{D} - \frac{A+C}{B+D} - k\left[\frac{AB+AD}{B(B+D)} + \frac{BC-AD}{B(B+D)}\right]$$

$$= \sum_{i=0}^{k-1} c_{x+i} + \frac{C}{D} - \frac{A+C}{B+D} - k\frac{A}{B} - k\frac{BC-AD}{B(B+D)}$$

$$= \left[\sum_{i=0}^{k-1} c_{x+i} - k\frac{A}{B}\right] + \left[\frac{C}{D} - \frac{A+C}{B+D} - k\frac{BC-AD}{B(B+D)}\right]$$

$$= \left[\sum_{i=0}^{k-1} c_{x+i} - k\frac{A}{B}\right] + \frac{CB(B+D) - (A+C)BD - k(BC-AD)D}{BD(B+D)}$$

$$= \left[\sum_{i=0}^{k-1} c_{x+i} - k\frac{A}{B}\right] + \frac{B^2C + BCD - ABD - BCD - kBCD + kAD^2}{BD(B+D)}$$

$$= \left[\sum_{i=0}^{k-1} c_{x+i} - k\frac{A}{B}\right] + \frac{B^2C - ABD + kAD^2 - kBCD}{BD(B+D)}$$

$$= \left[\sum_{i=0}^{k-1} c_{x+i} - k\frac{A}{B}\right] + \frac{B(BC-AD) + kD(AD-BC)}{BD(B+D)}$$

$$= \left[\sum_{i=0}^{k-1} c_{x+i} - k\frac{A}{B}\right] + \frac{(BC-AD)(B-kD)}{BD(B+D)}.$$

Now,

$$\left[\sum_{i=0}^{k-1} c_{x+i} - k\frac{A}{B}\right] > 0 \quad \text{because} \quad \sum_{i=0}^{k-1} \frac{C_{x+i}}{D_{x+i}} > k\frac{A}{B}$$

by the induction assumption that the lemma holds for $n = k$. Moreover,

$$vq_{x+k} = c_{x+k} = \frac{C_{x+k}}{D_{x+k}} > \frac{C_{x+k-1}}{D_{x+k-1}} > \frac{C_{x+k-2}}{D_{x+k-2}} > \cdots > \frac{C_x}{D_x}$$

since

$$q_x < q_{x+1} < q_{x+2} < \cdots < q_{x+k}.$$

Hence,

$$C_{x+k} = c_{x+k} \cdot D_{x+k},$$

and

$$C_{x+k-1} < c_{x+k} \cdot D_{x+k-1},$$

$$C_{x+k-2} < c_{x+k} \cdot D_{x+k-2},$$

$$\cdots$$

$$C_x < c_{x+k} \cdot D_x.$$

Adding, we have

$$M_x - M_{x+k} < c_{x+k}(N_x - N_{x+k})$$

or

$$\frac{M_x - M_{x+k}}{N_x - N_{x+k}} < c_{x+k} = \frac{C_{x+k}}{D_{x+k}}.$$

That is,

$$\frac{C}{D} > \frac{A}{B}$$

or

$$BC > AD$$

or

$$BC - AD > 0.$$

Also,

$$\begin{aligned} B - kD &= (N_x - N_{x+k}) - kD_{x+k} \\ &= (D_x - D_{x+k}) + (D_{x+1} - D_{x+k}) + (D_{x+2} - D_{x+k}) + \cdots \\ &\qquad + (D_{x+k-1} - D_{x+k}), \end{aligned}$$

which is greater than 0 because $D_x > D_{x+1} > D_{x+2} > \cdots > D_{x+k}$.

Since both $(B - kD)$ and $(BC - AD)$ are positive, and since

$$\left[\sum_{i=0}^{k-1} c_{x+1} - k\frac{A}{B}\right]$$

is positive,

$$\left[\sum_{i=0}^{k-1} c_{x+i} - k\frac{A}{B}\right] + \frac{(BC - AD)(B - kD)}{BD(B + D)}$$

must be positive. Hence,

$$\sum_{i=0}^{k} \frac{C_{x+i}}{D_{x+i}} - (k + 1)\frac{A + C_{x+k}}{B + D_{x+k}} > 0$$

so that tbe lemma holds for $n = k + 1$ if it holds for $n = k$.

It remains to complete the induction by showing that the lemma holds for $n = 2$.

$$\frac{C_x}{D_x} + \frac{C_{x+1}}{D_{x+1}} - 2\frac{C_x + C_{x+1}}{D_x + D_{x+1}}$$

$$= \frac{vd_x}{l_x} + \frac{vd_{x+1}}{l_{x+1}} - 2\frac{vd_x + v^2d_{x+1}}{l_x + vl_{x+1}}$$

$$= \frac{vd_xl_{x+1}(l_x + vl_{x+1}) + vd_{x+1}l_x(l_x + vl_{x+1}) - 2l_xl_{x+1}(vd_x + v^2d_{x+1})}{l_xl_{x+1}(l_x + vl_{x+1})}$$

$$= \frac{vd_xl_xl_{x+1} + v^2d_x(l_{x+1})^2 + vd_{x+1}(l_x)^2 + v^2d_{x+1}l_xl_{x+1} - 2vd_xl_xl_{x+1} - 2v^2d_{x+1}l_xl_{x+1}}{l_xl_{x+1}(l_x + vl_{x+1})}$$

$$= \frac{v^2d_x(l_{x+1})^2 - v^2d_{x+1}l_xl_{x+1} + vd_{x+1}(l_x)^2 - vd_xl_xl_{x+1}}{l_xl_{x+1}(l_x + vl_{x+1})}$$

$$= \frac{v^2l_{x+1}(d_xl_{x+1} - d_{x+1}l_x) + vl_x(d_{x+1}l_x - d_xl_{x+1})}{l_xl_{x+1}(l_x + vl_{x+1})}$$

$$= \frac{v(l_x - vl_{x+1})(d_{x+1}l_x - d_xl_{x+1})}{l_xl_{x+1}(l_x + vl_{x+1})}$$

$$= \frac{v(l_x - vl_{x+1})\left(\frac{d_{x+1}}{l_{x+1}} - \frac{d_x}{l_x}\right)}{l_x + vl_{x+1}}$$

$$= \frac{v(l_x - vl_{x+1})(q_{x+1} - q_x)}{l_x + vl_{x+1}}.$$

But $l_x > vl_{x+1}$ by the nature of the mortality table, and $q_{x+1} > q_x$ except for very young ages. Therefore, $\dfrac{v(l_x - vl_{x+1})(q_{x+1} - q_x)}{l_x + vl_{x+1}}$ must be positive so that

$$\frac{C_x}{D_x} + \frac{C_{x+1}}{D_{x+1}} > 2\frac{C_x + C_{x+1}}{D_x + D_{x+1}}$$

and the induction is complete, the lemma proved.

The theorem itself follows immediately from the lemma if we let $n = \omega - x + 1$ and observe that

$$c_x = \frac{C_x}{D_x} \quad \text{and} \quad P_x = \frac{M_x}{N_x} = \frac{C_x + C_{x+1} + C_{x+2} + \cdots + C_\omega}{D_x + D_{x+1} + D_{x+2} + \cdots + D_\omega}.$$

APPENDIX THREE

Tables

Table	Description
1	Compound Interest Functions
2	CSO Mortality Table
3	Commutation Columns
4	Single Premium Life Annuity, Single Premium Life Insurance, and Reciprocals
5	Valuation Factors and Natural Premiums
6	Single Premium Pure Endowment
7	Reciprocal of Single Premium Pure Endowment
8	Temporary Life Annuity Due
9	Reciprocal of Temporary Life Annuity Due
10	Single Premium Temporary Insurance
11	Foreborne Life Annuity Due
12	Accumulated Cost of Insurance
13	Net Level Annual Premiums—Ordinary Life and Twenty Pay Life
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All figures in Tables 1–13 are based on CSO mortality and 2½% interest. Table 14 is based on 1937 Standard Annuity mortality and 2½% interest.

TABLE 1

COMPOUND INTEREST FUNCTIONS—$2\frac{1}{2}$% INTEREST

n	$(1.025)^n$	v^n	$s_{\overline{n}\rceil}$	$a_{\overline{n}\rceil}$
1	1.025 000	.975 6098	1.000 000	0.975 610
2	1.050 625	.951 8144	2.025 000	1.927 424
3	1.076 891	.928 5994	3.075 625	2.856 024
4	1.103 813	.905 9506	4.152 516	3.761 974
5	1.131 408	.883 8543	5.256 329	4.645 828
6	1.159 693	.862 2969	6.387 737	5.508 125
7	1.188 686	.841 2652	7.547 430	6.349 391
8	1.218 403	.820 7466	8.736 116	7.170 137
9	1.248 863	.800 7284	9.954 519	7.970 866
10	1.280 085	.781 1984	11.203 382	8.752 064
11	1.312 087	.762 1448	12.483 466	9.514 209
12	1.344 889	.743 5559	13.795 553	10.257 765
13	1.378 511	.725 4204	15.140 442	10.983 185
14	1.412 974	.707 7272	16.518 953	11.690 912
15	1.448 298	.690 4656	17.931 927	12.381 378
16	1.484 506	.673 6249	19.380 225	13.055 003
17	1.521 618	.657 1951	20.864 730	13.712 198
18	1.559 659	.641 1659	22.386 349	14.353 364
19	1.598 650	.625 5277	23.946 007	14.978 891
20	1.638 616	.610 2709	25.544 658	15.589 162
21	1.679 582	.595 3863	27.183 274	16.184 549
22	1.721 571	.580 8647	28.862 856	16.765 413
23	1.764 611	.566 6972	30.584 427	17.332 110
24	1.808 726	.552 8754	32.349 038	17.884 986
25	1.853 944	.539 3906	34.157 764	18.424 376
26	1.900 293	.526 2347	36.011 708	18.950 611
27	1.947 800	.513 3997	37.912 001	19.464 011
28	1.996 495	.500 8778	39.859 801	19.964 889
29	2.046 407	.488 6613	41.856 296	20.453 550
30	2.097 568	.476 7427	43.902 703	20.930 293
31	2.150 007	.465 1148	46.000 271	21.395 407
32	2.203 757	.453 7706	48.150 278	21.849 178
33	2.258 851	.442 7030	50.354 034	22.291 881
34	2.315 322	.431 9053	52.612 885	22.723 786
35	2.373 205	.421 3711	54.928 207	23.145 157
36	2.432 535	.411 0937	57.301 413	23.556 251
37	2.493 349	.401 0670	59.733 948	23.957 318
38	2.555 682	.391 2849	62.227 297	24.348 603
39	2.619 574	.381 7414	64.782 979	24.730 344
40	2.685 064	.372 4306	67.402 554	25.102 775
41	2.752 190	.363 3470	70.087 617	25.466 122
42	2.820 995	.354 4848	72.839 808	25.820 607
43	2.891 520	.345 8389	75.660 803	26.166 446
44	2.963 808	.337 4038	78.552 323	26.503 849
45	3.037 903	.329 1744	81.516 131	26.833 024
46	3.113 851	.321 1458	84.554 034	27.154 170
47	3.191 697	.313 3129	87.667 885	27.467 483
48	3.271 490	.305 6712	90.859 582	27.773 154
49	3.353 277	.298 2158	94.131 072	28.071 369
50	3.437 109	.290 9422	97.484 349	28.362 312

TABLE 2

COMMISSIONERS 1941 STANDARD ORDINARY MORTALITY TABLE

x	l_x	d_x	$1000q_x$	$\mathring{e}_x$
0	1 023 102	23 102	22.58	62.33
1	1 000 000	5 770	5.77	62.76
2	994 230	4 116	4.14	62.12
3	990 114	3 347	3.38	61.37
4	986 767	2 950	2.99	60.58
5	983 817	2 715	2.76	59.76
6	981 102	2 561	2.61	58.92
7	978 541	2 417	2.47	58.08
8	976 124	2 255	2.31	57.22
9	973 869	2 065	2.12	56.35
10	971 804	1 914	1.97	55.47
11	969 890	1 852	1.91	54.58
12	968 038	1 859	1.92	53.68
13	966 179	1 913	1.98	52.78
14	964 266	1 996	2.07	51.89
15	962 270	2 069	2.15	50.99
16	960 201	2 103	2.19	50.10
17	958 098	2 156	2.25	49.21
18	955 942	2 199	2.30	48.32
19	953 743	2 260	2.37	47.43
20	951 483	2 312	2.43	46.54
21	949 171	2 382	2.51	45.66
22	946 789	2 452	2.59	44.77
23	944 337	2 531	2.68	43.88
24	941 806	2 609	2.77	43.00
25	939 197	2 705	2.88	42.12
26	936 492	2 800	2.99	41.24
27	933 692	2 904	3.11	40.36
28	930 788	3 025	3.25	39.49
29	927 763	3 154	3.40	38.61
30	924 609	3 292	3.56	37.74
31	921 317	3 437	3.73	36.88
32	917 880	3 598	3.92	36.01
33	914 282	3 767	4.12	35.15
34	910 515	3 961	4.35	34.29

TABLE 2—*Continued*

x	l_x	d_x	$1000q_x$	$\mathring{e}_x$
35	906 554	4 161	4.59	33.44
36	902 393	4 386	4.86	32.59
37	898 007	4 625	5.15	31.75
38	893 382	4 878	5.46	30.91
39	888 504	5 162	5.81	30.08
40	883 342	5 459	6.18	29.25
41	877 883	5 785	6.59	28.43
42	872 098	6 131	7.03	27.62
43	865 967	6 503	7.51	26.81
44	859 464	6 910	8.04	26.01
45	852 554	7 340	8.61	25.21
46	845 214	7 801	9.23	24.43
47	837 413	8 299	9.91	23.65
48	829 114	8 822	10.64	22.88
49	820 292	9 392	11.45	22.12
50	810 900	9 990	12.32	21.37
51	800 910	10 628	13.27	20.64
52	790 282	11 301	14.30	19.91
53	778 981	12 020	15.43	19.19
54	766 961	12 770	16.65	18.48
55	754 191	13 560	17.98	17.78
56	740 631	14 390	19.43	17.10
57	726 241	15 251	21.00	16.43
58	710 990	16 147	22.71	15.77
59	694 843	17 072	24.57	15.13
60	677 771	18 022	26.59	14.50
61	659 749	18 988	28.78	13.88
62	640 761	19 979	31.18	13.27
63	620 782	20 958	33.76	12.69
64	599 824	21 942	36.58	12.11
65	577 882	22 907	39.64	11.55
66	554 975	23 842	42.96	11.01
67	531 133	24 730	46.56	10.48
68	506 403	25 553	50.46	9.97
69	480 850	26 302	54.70	9.47

TABLE 2—*Continued*

x	l_x	d_x	$1000q_x$	$\mathring{e}_x$
70	454 548	26 955	59.30	8.99
71	427 593	27 481	64.27	8.52
72	400 112	27 872	69.66	8.08
73	372 240	28 104	75.50	7.64
74	344 136	28 154	81.81	7.23
75	315 982	28 009	88.64	6.82
76	287 973	27 651	96.02	6.44
77	260 322	27 071	103.99	6.07
78	233 251	26 262	112.59	5.72
79	206 989	25 224	121.86	5.38
80	181 765	23 966	131.85	5.06
81	157 799	22 502	142.60	4.75
82	135 297	20 857	154.16	4.46
83	114 440	19 062	166.57	4.18
84	95 378	17 157	179.88	3.91
85	78 221	15 185	194.13	3.66
86	63 036	13 198	209.37	3.42
87	49 838	11 245	225.63	3.19
88	38 593	9 378	243.00	2.98
89	29 215	7 638	261.44	2.77
90	21 577	6 063	280.99	2.58
91	15 514	4 681	301.73	2.39
92	10 833	3 506	323.64	2.21
93	7 327	2 540	346.66	2.03
94	4 787	1 776	371.00	1.84
95	3 011	1 193	396.21	1.63
96	1 818	813	447.19	1.37
97	1 005	551	548.26	1.08
98	454	329	724.67	0.78
99	125	125	1000.00	0.50

TABLE 3

Commutation Columns—CSO 2½%

x	D_x	N_x	S_x	C_x	M_x	R_x
0	1 023 102.00	81 374 229.80	784 658 955.65	22 538.536 6	257 876.883 9	12 236 206.496 7
1	975 609.76	30 351 127.80	753 284 725.85	5 491.969 1	235 338.347 3	11 976 329.612 8
2	946 622.43	29 375 518.04	722 933 598.05	8 822.115 2	229 846.378 2	11 742 991.265 5
3	919 419.26	26 429 195.61	693 558 080.01	3 032.216 6	226 024.263 0	11 513 144.887 3
4	893 962.20	27 509 776.33	665 128 884.40	2 607.370 2	222 992.046 2	11 287 120.624 3
5	869 550.88	26 615 814.13	637 619 108.07	2 341.136 0	220 384.676 0	11 064 126.578 1
6	846 001.18	25 746 283.25	611 003 293.94	2 154.480 3	216 043.540 0	10 843 743.902 1
7	823 212.53	24 900 262.07	585 257 030.69	1 983.744 5	215 889.059 7	10 625 700.362 1
8	801 150.42	24 077 049.54	560 356 768.62	1 805.642 5	213 905.315 2	10 409 811.302 4
9	779 804.53	23 275 899.12	536 279 719.06	1 613.174 7	212 099.672 7	10 195 905.987 2
10	759 171.76	22 496 094.59	513 003 819.96	1 458.745 1	210 466.496 0	9 983 806.314 5
11	739 196.60	21 736 922.86	490 507 725.87	1 377.065 5	209 027.752 9	9 773 319.816 5
12	719 790.36	20 997 726.26	468 770 802.51	1 848.556 5	207 650.887 4	9 564 292.063 6
18	700 885.94	20 277 935.90	447 773 076.25	1 353.882 1	206 302.130 9	9 356 641.376 2
14	682 437.26	19 577 049.96	427 495 140.35	1 876.169 3	204 948.248 6	9 150 339.245 8
15	664 414.29	18 894 612.66	407 916 090.39	1 393.730 0	203 570.079 5	6 945 390.996 5
16	646 615.33	18 230 198.39	389 023 477.71	1 382.081 2	202 176.349 5	8 741 620.917 0
17	629 657.27	17 583 383.06	370 793 279.32	1 382.353 7	200 794.266 3	6 539 644.567 5
18	612 917.42	16 953 725.79	353 209 896.26	1 375.535 5	199 411.914 6	6 338 650.299 2
19	596 592.68	16 340 808.37	330 256 170.47	1 379.212 3	196 036.379 1	6 139 438.384 6
20	580 662.42	15 744 215.69	319 915 362.10	1 376.533 1	196 657.166 8	7 941 402.005 5
21	565 123.40	15 163 553.27	304 171 146.41	1 383.619 6	195 280.633 7	7 744 744.838 7
22	549 956.28	14 598 429.87	289 007 593.14	1 389.541 6	193 897.014 1	7 549 464.205 0
23	535 153.17	14 048 473.59	274 409 163.27	1 399.327 5	192 507.472 5	7 355 567.190 9
24	520 701.32	13 513 320.42	260 360 689.68	1 407.270 0	191 108.145 0	7 163 059.718 4
25	506.594.02	12 992 619.10	246 847 369.26	1 423.464 9	189 700.675 0	6 971 951.573 4
26	492 814.61	12 486 025.08	233 854 750.16	1 487.519 2	188 277.410 1	6 782 250.698 4
27	479 357.22	11 993 210.47	221 388 725.08	1 454.549 1	186 839.890 9	6 593 973.288 3
28	466 211.03	11 513 853.25	209 375 514.61	1 478.200 3	185 385.341 8	0 407 133.397 4
29	453 361.83	11 047 642.22	197 661 661.36	1 503.646 4	183 907.141 5	6 221 748.055 6
30	440 800.58	10 594 280.39	186 614 019.14	1 531.156 0	182 403.495 1	6 037 840.914 1
31	426 518.16	10 153 479.81	176 219 736.75	1 559.609 4	180 872.337 1	5 855 437.419 0
62	416 506.91	9 724 961.63	166 066 258.94	1 592.845 3	179 312.727 7	5 674 565.081 9
33	404 755.37	9 308 454.72	156 341 297.31	1 626.967 4	177 719.882 4	5 495 252.354 2
34	893 256.29	8 903 699.35	147 032 842.59	1 669.050 6	176 092.695 0	5 317 532.471 8
35	381 995.63	8 510 443.06	136 129 143.24	1 710.561 0	174 423.844 2	5 141 439.576 8
36	370 968.10	8 128 447.43	129 618 700.16	1 759.080 1	172 713.283 2	4 967 015.732 6
37	360 161.02	7 757 479.33	121 490 252.75	1 809.692 6	170 954.208 1	4 794 302.449 4
38	349 566.90	7 397 318.31	113 732 773.42	1 862.134 5	169 144.510 3	4 623 346.246 3
39	339 178.75	7 047 751.41	106 335 455.11	1 922.486 9	167 262.375 6	4 454 203.736 0
40	328 983.61	6 708 572.66	99 267 703.70	1 983.511 0	165 359.888 9	4 286 921.860 2
41	318 976.11	6 379 589.05	92 579 131.04	2 050.694 7	163 376.377 9	4 121 561.471 3
42	309 145.51	6 060 612.94	86 199 541.99	2 120.336 1	161 325.683 2	3 958 185.093 4
43	299 485.04	5 751 467.43	80 136 929.05	2 194.136 7	159 205.345 1	3 796 859.410 2
44	289 986.39	5 451 982.39	74 387 461.62	2 274.595 1	157 011.208 4	3 637 654.065 1
45	280 638.95	5 161 996.00	68 935 479.23	2 357.209 9	154 736.613 3	3 480 642.856 7
46	271 436.89	4 881 357.05	63 773 483.23	2 444.154 2	152 379.403 4	3 325 906.248 4
47	262 372.36	4 609 920.16	56 692 126.16	2 536.765 0	149 935.249 2	6 173 526.840 0
46	253 436.24	4 347 547.83	54 282 206.02	2 630.659 4	147 398.484 2	3 023 591.590 8
49	244 624.00	4 094 111.59	49 934 658.19	2 782.529 2	144 767.624 8	2 876 193.106 6

TABLE 3—*Continued*

x	D_x	N_x	S_x	C_x	M_x	R_x
50	235 925.04	8 849 487.59	45 840 546.60	2 835.622 1	142 035.095 6	2 731 425.481 8
51	227 335.15	3 613 562.55	41 991 059.01	2 943.137 4	139 199.473 5	2 589 390.386 2
52	218 847.25	3 386 227.40	38 377 496.46	3 053.177 2	136 256.336 1	2 450 190.912 7
53	210 456.33	3 167 380.15	34 991 269.06	3 168.222 9	133 203.158 9	2 313 934.576 6
54	202 155.03	2 956 923.82	31 823 888.91	3 283.812 1	130 034.936 0	2 160 731.417 7
55	193 940.61	2 754 768.79	28 866 965.09	3 401.913 1	126 751.123 9	2 050 696.481 7
56	185 808.43	2 560 828.18	26 112 196.30	3 522.090 1	123 349.210 8	1 923 945.357 8
57	177 754.43	2 375 019.75	23 551 368.12	3 641.783 5	119 827.120 7	1 800 596.147 0
58	169 777.17	2 197 265.32	21 176 348.37	8 761.696 8	116 185.337 2	1 680 769.026 3
59	161 874.57	2 027 488.15	18 979 083.05	3 880.185 4	112 423.640 4	1 564 583.689 1
60	154 046.23	1 865 613.58	16 951 594.90	3 996.199 9	108 543.455 0	1 452 160.048 7
61	146 292.60	1 711 567.35	15 085 981.32	4 107.708 0	104 547.255 1	1 343 616.593 7
62	138 616.97	1 565 274.55	13 374 413.97	4 216.676 0	100 439.547 1	1 239 069.338 6
63	131 019.40	1 426 657.58	11 809 139.42	4 315.413 8	96 222.871 1	1 138 629.791 5
64	123 508.39	1 295 638.18	10 382 481.84	4 407.831 2	91 907.457 3	1 042 406.920 4
65	116 088.15	1 172 129.79	9 086 843.66	4 489.449 7	87 499.626 1	950 499.463 1
66	108 767.29	1 058 041.64	7 914 713.87	4 558.728 2	83 010.176 4	862 999.837 0
67	101 555.70	947 274.35	6 858 672.23	4 613.189 3	78 451.448 2	779 989.660 6
68	94 465.545	845 718.651	5 911 397.883	4 650.452 1	73 838.258 9	701 538.212 4
69	87 511.050	751 253.106	5 065 679.232	4 670.014 3	69 187.806 8	627 699.953 5
70	80 706.625	663 742.056	4 314 426.126	4 669.226 0	64 517.792 5	558 512.146 7
71	74 068.942	583 035.431	3 650 684.070	4 644.235 4	59 848.566 5	493 994.354 2
72	67 618.148	508 966.489	3 067 648.639	4 595.428 1	55 204.331 1	434 145.787 7
73	61 373.498	441 348.341	2 558 682.150	4 520.662 7	50 608.903 0	378 941.456 6
74	55 355.921	379 974.843	2 117 333.809	4 418.249 2	46 088.240 3	328 332.553 6
75	49 587.526	324 618.922	1 737 358.966	4 288.286 9	41 669.991 1	282 244.313 3
76	44 089.787	275 031.396	1 412 740.044	4 130.220 2	37 381.704 2	240 574.322 2
77	38 884.206	230 941.609	1 137 708.648	3 944.961 8	33 251.484 0	203 192.618 0
78	33 990.850	192 057.403	906 767.039	3 733.725 8	29 306.522 2	169 941.134 0
79	29 428.077	158 086.553	714 709.636	3 498.684 1	25 572.796 4	140 634.611 8
80	25 211.636	128 638.476	556 643.083	3 243.115 8	22 074.112 3	115 061.815 4
81	21 353.602	103 426.840	428 004.607	2 970.736 8	18 830.996 5	92 987 703 1
82	17 862.047	82 073.238	324 577.767	2 686.402 0	15 860.259 7	74 156.706 6
83	14 739.984	84 211.191	242 504.529	2 395.321 2	13 173.857 7	58 296.446 9
84	11 985.151	49 471.207	178 293.338	2 103.356 1	10 778.536 5	45 122.589 2
85	9 589.474 6	37 486.056 1	128 822.130 6	1 816.194 6	8 675.180 4	34 344.052 7
86	7 539.390 5	27 896.581 5	91 336.074 5	1 540.039 4	6 858.985 8	25 668.872 3
87	5 815.463 2	20 357.191 0	63 439.493 0	1 280.145 4	5 318.946 4	18 809.886 5
88	4 393.477 3	14 541.727 8	43 082.302 0	1 041.564 6	4 038.601 0	13 490.940 1
89	3 244.754 6	10 148.250 5	28 540.574 2	827.621 5	2 997.236 4	9 452.139 1
90	2 337.992 9	6 903.495 9	18 392.323 7	640.937 7	2 169.614 9	6 454.902 7
91	1 640.030 9	4 565.503 0	11 488.827 8	482.773 0	1 528.677 2	4 285.287 8
92	1 117.257 1	2 925.472 1	6 923.324 8	352.770 7	1 045.904 2	2 756.610 6
93	737.236 3	1 608.215 0	3 997.852 7	249.339 1	693.133 5	1 710.706 4
94	469.915 8	1 070.978 7	2 189.637 7	170.088 8	443.794 4	1 017.572 9
95	288.365 7	601.062 9	1 118.659 0	111.467 8	273.705 6	573.778 5
96	169.864 6	312.697 2	517.596 1	74.109 8	162.237 8	300.072 9
97	91.611 7	142.832 6	204.898 9	49.001 9	88.128 0	137.835 1
98	40.375 5	51.220 9	62.066 3	28.545 1	39.126 1	49.707 1
99	10.845 4	10.845 4	10.845 4	10.581 0	10.581 0	10.581 0

TABLE 4

SINGLE-PREMIUM LIFE ANNUITY, SINGLE-PREMIUM LIFE INSURANCE, AND RECIPROCALS—CSO 2½%

x	$\ddot{a}_x$	$\frac{1000}{\ddot{a}_x}$	$\frac{1000}{a_x}$	$1000A_x$	$\frac{1}{A_x}$
20	27.114 232	36.881 00	38.293 30	338.677 27	2.952 663
21	26.832 288	37.268 53	38.711 24	345.553 95	2.893 904
22	26.544 710	37.672 29	39.147 05	352.568 05	2.836 332
23	26.251 314	38.093 33	39.601 90	359.724 05	2.779 909
24	25.952 153	38.532 45	40.076 70	367.020 66	2.724 642
25	25.647 005	38.990 91	40.572 88	374.463 29	2.670 489
26	25.336 151	39.469 29	41.091 13	382.045 10	2.617 492
27	25.019 360	39.969 05	41.633 08	389.771 70	2.565 604
28	24.696 656	40.491 31	42.200 05	397.642 53	2.514 822
29	24.368 268	41.036 97	42.793 07	405.652 00	2.465 167
30	24.034 180	41.607 41	43.413 74	413.800 49	2.416 624
31	23.694 397	42.204 07	44.063 74	422.087 88	2.369 175
32	23.348 860	42.828 64	44.745 01	430.515 61	2.322 796
33	22.997 731	43.482 55	45.459 23	439.079 73	2.277 491
34	22.640 958	44.167 74	46.208 68	447.781 51	2.233 232
35	22.278 902	44.885 52	46.994 91	456.612 14	2.190 043
36	21.911 446	45.638 25	47.820 70	465.574 49	2.147 884
37	21.538 920	46.427 58	48.688 05	474.660 49	2.106 769
38	21.161 380	47.255 90	49.599 78	483.868 78	2.066 676
39	20.778 871	48.125 81	50.559 00	493.198 27	2.027 582
40	20.391 814	49.039 29	51.568 15	502.638 68	1.989 501
41	20.000 209	49.999 48	52.631 00	512.190 02	1.952 400
42	19.604 402	51.008 95	53.750 72	521.843 85	1.916 282
43	19.204 523	52.071 07	54.931 40	531.597 00	1.881 124
44	18.800 822	53.189 16	56.177 18	541.443 36	1.846 915
45	18.393 726	54.366 36	57.491 99	551.372 53	1.813 656
46	17.983 396	55.606 85	58.881 04	561.380 58	1.781 323
47	17.570 146	56.914 72	60.349 50	571.459 85	1.749 904
48	17.154 405	58.294 06	61.902 62	581.599 88	1.719 395
49	16.736 344	59.750 21	63.547 16	591.796 49	1.689 770
50	16.316 571	61.287 39	65.288 76	602.034 85	1.661 033
51	15.895 310	62.911 64	67.135 23	612.309 51	1.633 161
52	15.473 018	64.628 63	69.094 09	622.609 32	1.606 144
53	15.050 059	66.444 92	71.174 08	632.925 39	1.579 965
54	14.627 011	68.366 67	73.383 66	643.243 63	1.554 621
55	14.204 188	70.401 77	75.733 55	653.556 39	1.530 090
56	13.782 088	72.557 95	78.234 48	663.851 51	1.506 361
57	13.361 241	74.843 35	80.898 03	674.116 07	1.483 424
58	12.942 054	77.267 49	83.737 69	684.340 15	1.461 262
59	12.525 056	79.839 96	86.767 47	694.510 83	1.439 862

TABLE 4—*Continued*

x	$\ddot{a}_x$	$\frac{1000}{\ddot{a}_x}$	$\frac{1000}{a_x}$	$1000A_x$	$\frac{1}{A_x}$
60	12.110 739	82.571 34	90.003 01	704.616 12	1.419 212
61	11.699 601	85.473 00	93.461 43	714.643 88	1.399 298
62	11.292 084	88.557 61	97.162 05	724.583 32	1.380 104
63	10.888 904	91.836 61	101.123 44	734.416 97	1.361 624
64	10.490 285	95.326 29	105.370 91	744.139 39	1.343 834
65	10.096 894	99.040 36	109.927 63	753.734 29	1.326 727
66	9.709 184	102.995 27	114.821 32	763.190 63	1.310 289
67	9.327 633	107.208 33	120.082 14	772.496 76	1.294 504
68	8.952 668	111.698 55	125.743 96	781.642 24	1.279 358
69	8.584 665	116.486 78	131.844 98	790.617 93	1.264 833
70	8.224 134	121.593 35	138.424 90	799.411 37	1.250 920
71	7.871 524	127.040 20	145.528 12	808.011 61	1.237 606
72	7.527 069	132.853 84	153.208 12	816.412 95	1.224 870
73	7.191 188	139.059 08	161.519 89	824.605 17	1.212 702
74	6.864 213	145.683 12	170.525 87	832.580 17	1.201 086
75	6.546 383	152.756 11	180.297 68	840.332 12	1.190 006
76	6.237 984	160.308 20	190.913 15	847.854 05	1.179 448
77	5.939 214	168.372 45	202.461 36	855.141 12	1.169 398
78	5.650 268	176.982 76	215.041 37	862.188 58	1.159 839
79	5.371 284	186.175 22	228.765 74	868.993 07	1.150 757
80	5.102 345	195.988 32	243.763 02	875.552 56	1.142 136
81	4.843 531	206.460 95	260.177 43	881.865 10	1.133 960
82	4.594 840	217.635 43	278.176 50	887.930 73	1.126 214
83	4.356 259	229.554 76	297.950 78	893.749 78	1.118 881
84	4.127 708	242.265 20	319.722 94	899.324 19	1.111 946
85	3.909 083	255.814 47	343.750 93	904.656 51	1.105 392
86	3.700 111	270.262 16	370.355 14	909.753 39	1.099 199
87	3.500 528	285.671 19	399.915 54	914.621 27	1.093 349
88	3.309 845	302.128 98	432.929 48	919.272 07	1.087 817
89	3.127 586	319.735 41	470.016 25	923.717 41	1.082 582
90	2.952 745	338.667 92	512.099 63	927.981 83	1.077 607
91	2.783 791	359.222 37	560.603 79	932.102 66	1.072 843
92	2.618 441	381.906 64	617.878 56	936.135 59	1.068 221
93	2.452 694	407.714 95	688.376 22	940.178 19	1.063 628
94	2.279 086	438.772 39	781.808 26	944.412 54	1.058 859
95	2.084 377	479.759 66	922.188 50	949.161 54	1.053 561
96	1.840 862	543.223 77	1 189.255 79	955.100 93	1.047 010
97	1.559 108	641.392 39	1 788.563 21	961.972 98	1.039 530
98	1.268 615	788.261 21	3 722.800 29	969.058 17	1.031 930
99	1.000 000	1 000.000 00		975.609 76	1.025 000

TABLE 5

VALUATION FACTORS AND NATURAL PREMIUMS—CSO 2½%

x	u_x	$1000k_x$	$1000c_x$	x	u_x	$1000k_x$	$1000c_x$
20	1.027 496 7	2.435 81	2.370 63	60	1.052 999 4	27.316 45	25.941 57
21	1.027 578 8	2.515 87	2.448 35	61	1.055 374 4	29.633 51	28.078 68
22	1.027 661 4	2.596 53	2.526 64	62	1.057 988 1	32.183 60	30.419 63
23	1.027 754 6	2.687 39	2.614 82	63	1.060 813 8	34.940 25	32.937 22
24	1.027 847 3	2.777 90	2.702 65	64	1.063 919 0	37.969 69	35.688 52
25	1.027 960 6	2.888 44	2.809 88	65	1.067 307 6	41.275 73	38.672 77
26	1.028 073 8	2.998 85	2.916 96	66	1.071 011 2	44.888 95	41.912 68
27	1.028 197 9	3.119 94	3.034 38	67	1.075 055 5	48.834 62	45.425 22
28	1.028 342 0	3.260 53	3.170 67	68	1.079 469 9	53.141 31	49.229 09
29	1.028 496 4	3.411 17	3.316 66	69	1.084 310 6	57.864 08	53.364 85
30	1.028 662 5	3.573 15	3.473 59	70	1.089 614 9	63.038 92	57.854 31
31	1.028 838 1	3.744 50	3.639 54	71	1.095 400 3	68.683 27	62.701 53
32	1.029 033 7	3.935 33	3.824 30	72	1.101 748 3	74.876 42	67.961 46
33	1.029 240 7	4.137 22	4.019 68	73	1.108 707 0	81.665 39	73.658 23
34	1.029 478 5	4.369 29	4.244 18	74	1.116 327 5	89.100 01	79.815 30
35	1.029 726 4	4.611 07	4.477 96	75	1.124 694 2	97.262 59	86.479 15
36	1.030 006 2	4.884 15	4.741 87	76	1.133 873 9	106.218 45	93.677 49
37	1.030 306 4	5.176 96	5.024 68	77	1.143 961 0	116.059 52	101.454 09
38	1.030 627 4	5.490 13	5.326 98	78	1.155 048 3	126.876 31	109.845 04
39	1.030 989 8	5.843 72	5.668 07	79	1.167 241 9	138.772 59	118.889 32
40	1.031 373 8	6.218 37	6.029 21	80	1.180 673 7	151.876 75	128.635 68
41	1.031 799 3	6.633 43	6.429 00	81	1.195 473 4	166.315 58	139.121 11
42	1.032 256 9	7.079 95	6.858 71	82	1.211 809 1	182.252 71	150.397 22
43	1.032 755 5	7.566 34	7.326 37	83	1.229 853 8	199.857 41	162.505 01
44	1.033 307 7	8.105 06	7.843 80	84	1.249 823 5	219.340 08	175.496 83
45	1.033 901 3	8.684 19	8.399 44	85	1.271 916 4	240.894 09	189.394 59
46	1.034 548 5	9.315 59	9.004 51	86	1.296 438 5	264.818 00	204.265 78
47	1.035 259 7	10.009 48	9.668 57	87	1.323 658 4	291.374 09	220.127 85
48	1.036 023 6	10.754 71	10.380 76	88	1.354 024 5	320.999 48	237.070 67
49	1.036 871 7	11.582 19	11.170 33	89	1.387 837 7	353.988 05	255.064 43
50	1.037 785 1	12.473 31	12.019 17	90	1.425 578 6	390.808 32	274.140 15
51	1.038 784 6	13.448 36	12.946 25	91	1.467 908 2	432.105 61	294.368 27
52	1.039 870 1	14.507 41	13.951 18	92	1.515 466 8	478.504 17	315.747 06
53	1.041 064 0	15.672 24	15.054 07	93	1.568 868 7	530.603 72	338.207 84
54	1.042 355 3	16.932 05	16.244 03	94	1.629 583 2	589.837 28	361.955 91
55	1.043 766 5	18.308 71	17.541 01	95	1.697 621 0	656.215 61	386.550 13
56	1.045 309 7	19.814 36	18.955 50	96	1.854 179 2	808.955 23	436.287 54
57	1.046 986 6	21.450 37	20.487 73	97	2.268 997 9	1 213.656 38	534.886 55
58	1.048 819 3	23.238 34	22.156 67	98	3.722 800 0	2 631.999 97	706.994 74
59	1.050 818 1	25.188 45	23.970 32	99			975.609 76

TABLE 6

Single-Premium Pure Endowment—CSO 2½%

$1000{}_nE_x$

x	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 15$	$n = 20$	To Age 60 $n = 60 - x$	To Age 65 $n = 65 - x$
20	973.239 13	947.118 76	921.625 28	696.736 73	872.441 55	759.133 98	657.861 62	568.566 06	265.293 94	199.923 57
21	973.161 41	946.966 94	921.394 04	896.430 82	872.047 79	758.273 66	656.437 35	564.436 25	272.588 65	205.420 91
22	973.083 12	946.804 95	921.153 27	696.097 79	871.627 87	757.345 51	654.890 27	562.127 44	280.106 31	211.085 16
23	972.994 94	946.633 70	920.885 15	895.738 35	871.173 06	756.335 55	653.209 09	559.524 90	267.854 45	216.925 11
24	972.907 11	946.443 92	920.599 19	895.352 10	870.675 38	755.243 50	651.388 32	556.915 03	295.843 73	222.945 77
25	972.799 88	946.235 44	920.285 29	694.921 39	870.125 90	754.046 88	649.402 88	553.972 10	304.082 20	229.154 22
26	972.592 80	946.017 07	919.943 98	894.455 19	869.532 23	752.753 68	647.253 80	550.789 08	912.564 54	235.591 52
27	972.575 36	945.770 32	919.565 96	893.943 32	868.886 30	751.341 60	644.916 79	547.341 98	321.359 99	242.174 63
28	972.439 09	945.495 82	919.150 68	893.367 10	868.180 59	749.804 04	642.360 85	543.606 43	330.421 67	249.003 45
29	972.293 10	945.201 29	916.707 51	892.786 60	867.422 60	746.141 41	639.635 67	539.577 88	339.786 50	256.060 72
30	972.136 17	944.887 41	918.227 85	892.141 06	866.595 13	746.332 09	636.657 41	535.219 44	349.469 21	263.357 54
31	971.970 22	944.546 53	917.712 03	891.433 89	865.699 79	744.370 09	633.431 49	530.514 58	359.485 86	270.906 02
32	971.785 46	944.177 11	917.141 15	890.664 93	664.717 98	742.233 81	629.935 11	525.434 85	369.852 75	278.718 44
33	971.590 08	943.769 16	916.524 24	889.823 95	663.649 86	739.916 17	626.146 72	519.959 35	380.590 95	286.810 67
34	971.365 58	943.324 00	915.842 98	888.903 54	862.487 80	737.397 97	622.047 28	514.054 14	391.719 57	295.197 20
35	971.131 80	942.840 68	915.107 10	887.912 96	861.223 51	734.665 36	617.611 88	507.703 73	403.266 99	303.899 18
36	970.867 89	942.309 99	914.307 06	886.824 54	859.847 83	731.698 75	612.815 89	500.874 42	415.254 64	312.932 99
37	970.585 08	941.741 99	913.434 82	885.648 64	858.353 61	728.486 19	607.637 24	493.541 57	427.714 88	322.322 94
38	970.282 78	941.117 72	912.489 44	884.367 20	856.731 65	725.000 66	602.048 79	485.678 61	440.677 37	332.091 36
39	969.941 69	940.436 60	911.453 05	882.971 10	854.968 26	721.224 45	596.013 24	477.254 45	454.174 16	342.262 47
40	969.580 55	939.699 91	910.334 21	881.461 50	853.046 43	717.133 10	589.514 49	468.246 93	488.248 93	352.869 12
41	969.180 76	936.894 88	909.116 32	879.611 82	850.963 08	712.702 73	582.515 21	456.632 46	462.939 59	363.939 97
42	968.751 05	938.025 55	907.789 19	878.023 08	648.701 72	707.910 16	574.966 29	448.387 48	498.296 62	375.513 00
43	968.283 39	937.071 70	906.345 42	676.076 25	646.240 07	702.727 37	566.897 00	437.492 27	514.370 35	367.625 90
44	967.765 96	936.033 22	904.774 63	873.959 09	843.570 66	697.119 03	558.214 39	425.910 98	581.219 61	400.322 78
45	967.210 32	934.910 57	903.088 64	671.688 04	840.671 03	691.068 01	548.912 49	413.656 60	549.912 49	413.656 60
46	966.605 25	933.683 83	901.218 71	869.170 86	837.524 86	884.536 41	538.956 95	400.709 29	567.521 32	427.680 09
47	965.941 19	932.354 45	699.199 40	666.460 06	834.109 50	677.489 24	528.321 63	387.097 11	587.126 32	442.455 79
48	965.229 00	930.904 91	897.011 21	883.519 96	830.411 37	689.900 93	516.971 82	372.738 88	607.830 30	458.056 65
49	964.439 43	929.324 76	694.627 04	860.325 76	826.390 80	661.728 08	504.690 72	357.736 96	629.726 52	474.557 49
50	963.590 59	927.613 50	892.047 48	856.861 28	822.043 34	652.945 64	492.055 29	342.085 88	652.945 64	492.055 26
51	962.663 51	925.753 62	889.237 91	853.104 37	617.332 65	643.511 59	478.444 66	325.613 88	977.617 29	510.647 64
52	961.658 58	923.726 92	886.191 66	849.032 54	612.230 59	633.396 02	464.046 33	306.974 16	703.699 39	530.452 89
53	960.555 69	921.524 21	882.883 55	844.614 30	806.709 73	622.549 07	448.860 53	291.621 05	731.962 89	551.602 10
54	959.365 73	919.138 33	670.297 59	839.836 50	800.744 71	610.958 77	432.690 80	273.829 05	762.020 24	574.253 12
55	958.068 75	916.540 55	675.406 07	834.660 53	794.295 87	598.575 61	416.140 91	255.684 07	794.295 67	596.575 61
56	956.654 26	913.721 56	871.190 64	629.059 37	787.331 32	585.373 25	396.630 69	237.286 26	929.059 37	624.773 34
57	955.122 03	910.664 05	868.623 82	823.005 08	779.822 90	571.325 64	380.402 03	218.752 36	966.623 82	653.091 65
58	953.453 09	907.343 58	661.675 32	616.464 15	771.713 85	556.408 97	361.494 39	200.208 59	907.343 56	683.797 76
59	951.639 44	903.741 70	856.323 36	809.388 36	762.988 20	540.610 25	341.967 99	181.795 55	951.639 44	717.148 82
60	949.668 19	899.840 13	850.520 00	801.761 85	753.593 00	523.911 71	321.900 29	163.662 89	1 000.000 00	753.593 00

TABLE 7

Reciprocal of Single-Premium Pure Endowment—CSO 2½%

$$\frac{1}{{}_nE_x}$$

x	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 15$	$n = 20$	To Age 60 $n = 60 - x$	To Age 65 $n = 65 - x$
20	1.027 497	1.055 834	1.085 040	1.115 155	1.145 209	1.317 291	1.520 076	1.765 019	3.760 404	5.001 909
21	1.027 579	1.056 003	1.085 312	1.115 535	1.146 725	1.318 785	1.523 375	1.771 679	3.668 531	4.868 054
22	1.027 661	1.056 184	1.086 596	1.115 950	1.147 279	1.320 401	1.526 973	1.778 958	3.570 073	4.737 402
23	1.027 755	1.056 375	1.085 912	1.116 397	1.147 878	1.322 164	1.530 903	1.786 911	3.473 978	4.609 886
24	1.027 847	1.056 587	1.086 249	1.116 879	1.148 534	1.324 076	1.535 183	1.795 606	3.380 163	4.485 390
25	1.027 961	1.056 819	1.086 820	1.117 417	1.149 259	1.326 177	1.539 876	1.805 145	3.288 584	4.363 873
26	1.028 074	1.057 063	1.087 023	1.117 999	1.150 044	1.328 455	1.544 989	1.815 577	3.199 135	4.245 178
27	1.028 198	1.057 339	1.087 470	1.118 639	1.150 899	1.330 953	1.550 566	1.827 011	3.111 775	4.129 252
28	1.028 342	1.057 646	1.087 961	1.119 336	1.151 834	1.333 682	1.558 709	1.839 559	3.026 436	4.016 009
29	1.028 496	1.057 975	1.088 466	1.120 088	1.152 841	1.336 646	1.563 390	1.853 301	2.943 025	3.905 324
30	1.028 662	1.058 327	1.089 054	1.120 899	1.153 941	1.339 886	1.570 703	1.868 393	9.861 482	8.797 119
31	1.028 838	1.058 709	1.089 666	1.121 788	1.155 135	1.343 418	1.578 703	1.884 962	2.781 751	3.691 317
82	1.029 034	1.059 123	1.090 345	1.122 757	1.156 446	1.347 284	1.587 485	1.903 186	2.703 779	8.587 850
33	1.029 241	1.059 581	1.091 079	1.123 818	1.157 677	1.351 504	1.597 070	1.923 227	2.627 493	3.486 621
34	1.029 479	1.060 081	1.091 890	1.124 981	1.159 437	1.356 120	1.607 595	1.945 320	2.552 846	3.887 566
35	1.029 726	1.060 625	1.092 766	1.128 237	1.161 139	1.361 164	1.619 140	1.969 653	2.479 747	3.290 565
86	1.030 008	1.081 222	1.093 724	1.127 619	1.182 998	1.366 683	1.631 811	1.998 508	2.408 181	3.195 572
37	1.030 308	1.061 662	1.094 760	1.129 116	1.165 021	1.372 710	1.645 719	2.026 172	2.338 006	3.102 479
38	1.030 627	1.062 566	1.095 903	1.130 752	1.157 227	1.379 309	1.660 995	2.058 975	9.269 234	3.011 219
39	1.030 990	1.063 338	1.097 149	1.132 540	1.189 637	1.386 531	1.677 815	2.095 318	2.201 799	2.921 734
40	1.031 374	1.064 171	1.098 498	1.134 479	1.172 266	1.394 441	1.696 311	2.135 818	2.135 515	8.833 912
41	1.031 799	1.065 082	1.090 969	1.136 607	1.175 139	1.403 110	1.716 693	2.180 395	2.070 652	2.747 706
42	1.082 257	1.066 069	1.101 577	1.138 922	1.178 270	1.412 609	1.739 172	2.230 214	8.006 835	2.663 024
43	1.032 756	1.067 154	1.103 332	1.141 451	1.181 698	1.423 027	1.763 989	2.285 807	1.944 125	2.579 807
44	1.033 308	1.068 338	1.105 248	1.144 218	1.185 437	1.434 475	1.791 426	2.347 908	1.882 463	2.497 984
45	1.033 901	1.080 821	1.107 336	1.147 226	1.189 528	1.447 036	1.821 784	2.417 464	1.821 784	2.417 464
46	1.034 548	1.071 026	1.109 609	1.150 522	1.193 994	1.460 843	1.855 436	2.495 575	1.762 048	2.338 196
47	1.035 260	1.072 553	1.112 100	1.154 121	1.198 883	1.476 038	1.892 786	2.583 531	1.703 205	2.260 113
48	1.036 024	1.074 224	1.114 818	1.158 051	1.204 222	1.492 758	1.934 341	2.682 843	1.645 196	2.183 136
49	1.036 872	1.076 050	1.117 784	1.162 350	1.210 081	1.511 195	1.980 627	2.795 350	1.587 991	2.107 226
50	1.037 785	1.078 035	1.121 017	1.187 050	1.216 481	1.531 521	2.032 292	2.923 243	1.531 521	2.032 292
51	1.038 785	1.080 201	1.124 558	1.172 190	1.223 492	1.553 974	2.090 106	3.069 237	1.475 759	1.958 298
52	1.039 870	1.082 571	1.128 424	1.177 811	1.231 177	1.578 791	2.154 948	3.236 517	1.420 660	1.885 182
53	1.041 064	1.085 159	1.132 652	1.183 972	1.239 603	1.606 299	2.227 864	3.429 108	1.366 189	1.812 901
54	1.042 355	1.087 976	1.137 271	1.190 708	1.248 837	1.638 772	2.310 051	3.651 913	1.312 301	1.741 392
55	1.043 766	1.091 050	1.142 324	1.198 092	1.258 977	1.670 632	2.403 032	3.911 077	1.258 977	1.870 632
56	1.045 810	1.094 425	1.147 854	1.206 186	1.270 113	1.708 312	2.508 588	4.214 319	1.206 186	1.600 580
57	1.045 987	1.098 100	1.153 903	1.215 059	1.282 342	1.750 315	2.828 798	4.571 379	1.153 903	1.531 202
58	1.048 819	1.102 116	1.160 530	1.224 794	1.295 617	1.797 239	2.766 295	4.994 791	1.102 118	1.462 485
59	1.050 818	1.108 511	1.167 783	1.235 501	1.810 638	1.849 761	2.924 250	5.500 685	1.050 818	1.394 411
60	1.052 999	1.111 309	1.175 751	1.247 253	1.326 976	1.908 719	3.106 552	6.110 124	1.000 000	1.326 975

TABLE 8

Temporary Life Annuity Due—CSO 2½%

$\ddot{a}_{x:\overline{n}|}$

x	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 15$	$n = 20$	To Age 60 $n = 60 - x$	To Age 65 $n = 65 - x$
20	1.000 000	1.973 239	2.920 858	3.841 983	4.738 720	8.869 069	12.457 794	15.560 923	23.901 827	25.095 624
21	1.000 000	1.973 161	2.920 128	3.841 522	4.737 953	8.865 450	12.448 796	15.543 445	23.531 038	24.758 175
22	1.000 000	1.973 083	2.919 888	3.841 041	4.737 139	8.861 556	12.439 081	15.524 538	23.152 418	24.413 396
23	1.000 000	1.972 995	2.919 629	3.840 514	4.735 252	8.857 313	12.428 508	15.503 984	22.766 183	24.061 044
24	1.000 000	1.972 907	2.919 351	3.839 950	4.735 302	8.852 718	12.417 040	15.481 693	22.369 267	23.701 093
25	1.000 000	1.972 800	2.919 035	3.839 321	4.734 242	8.847 669	12.404 502	15.457 394	21.964 244	23.333 259
26	1.000 000	1.972 893	2.918 710	3.838 654	4.738 109	8.842 225	12.390 940	15.431 093	21.550 521	22.957 711
27	1.000 000	1.972 575	2.918 346	3.837 912	4.731 855	8.838 273	12.378 152	15.402 481	21.127 453	22.574 148
28	1.000 000	1.972 439	2.917 935	3.837 086	4.730 473	8.829 767	12.360 038	15.371 377	20.695 005	22.182 494
29	1.000 000	1.972 293	2.917 494	3.836 202	4.728 989	8.822 734	12.342 592	15.337 707	20.253 202	21.782 850
30	1.000 000	1.972 138	2.917 024	3.835 251	4.727 392	8.815 115	12.323 678	15.301 234	19.801 850	21.375 087
31	1.000 000	1.971 970	2.916 517	3.834 229	4.725 663	8.808 839	12.303 148	15.281 703	19.340 758	20.959 087
32	1.000 000	1.971 785	2.915 963	3.833 104	4.723 769	8.797 810	12.280 808	15.218 797	18.869 670	20.534 670
33	1.000 000	1.971 590	2.915 359	3.831 883	4.721 707	8.787 993	12.258 558	15.172 312	18.388 493	20.101 834
34	1.000 000	1.971 368	2.914 690	3.830 533	4.719 438	8.777 271	12.230 161	15.121 883	17.896 944	19.660 384
35	1.000 000	1.971 132	2.913 972	3.829 080	4.716 992	8.765 669	12.201 594	15.067 383	17.395 041	19.210 464
36	1.000 000	1.970 868	2.913 178	3.827 485	4.714 309	8.753 018	12.170 548	15.008 350	16.882 405	18.751 795
37	1.000 000	1.970 585	2.912 327	3.825 762	4.711 411	8.739 311	12.138 938	14.944 592	18.358 977	18.284 450
38	1.000 000	1.970 283	2.911 401	3.823 890	4.708 257	8.724 426	12.100 511	14.875 702	15.824 452	17.808 289
39	1.000 000	1.969 942	2.910 378	3.821 831	4.704 802	8.708 210	12.060 978	14.801 232	15.278 486	17.823 083
40	1.000 000	1.969 581	2.909 279	3.819 614	4.701 075	8.690 661	12.018 240	14.720 974	14.720 974	16.828 932
41	1.000 000	1.969 181	2.908 078	3.817 192	4.697 004	8.871 579	11.971 934	14.634 393	14.151 453	18.325 546
42	1.000 000	1.968 751	2.908 777	3.814 568	4.892 589	8.650 898	11.921 872	14.541 173	18.569 660	15.812 887
43	1.000 000	1.968 283	2.905 355	3.811 701	4.687 779	8.828 435	11.867 712	14.440 821	12.975 119	15.290 706
44	1.000 000	1.987 766	2.903 799	3.808 574	4.682 533	8.604 054	11.809 155	14.832 894	12.367 370	14.758 805
45	1.000 000	1.967 210	2.902 121	3.805 190	4.878 858	8.577 666	11.745 990	14.217 079	11.745 990	14.217 079
46	1.000 000	1.966 605	2.900 289	3.801 508	4.870 879	8.549 055	11.877 815	14.092 835	11.110 293	13.665 155
47	1.000 000	1.965 941	2.898 298	3.797 495	4.663 955	8.518 049	11.604 294	13.959 726	10.459 589	13.102 717
48	1.000 000	1.965 229	2.896 134	3.793 145	4.656 665	8.484 511	11.525 148	13.817 397	9.793 131	12.529 455
49	1.000 000	1.964 439	2.893 764	3.788 391	4.648 717	8.448 163	11.439 897	13.665 292	9.109 891	11.944 788
50	1.000 000	1.963 591	2.891 204	3.783 252	4.640 113	8.408 917	11.348 341	13.503 211	8.408 917	11.348 341
51	1.000 000	1.962 664	2.888 417	3.777 655	4.630 759	8.366 481	11.250 002	13.330 658	7.688 864	10.739 355
52	1.000 000	1.961 659	2.885 385	3.771 577	4.820 609	8.320 657	11.144 545	13.147 348	8.948 289	10.117 091
53	1.000 000	1.960 556	2.882 080	3.764 963	4.609 578	8.271 182	11.031 559	12.952 957	6.185 447	9.480 591
54	1.000 000	1.959 366	2.878 504	3.757 802	4.597 638	8.217 879	10.910 788	12.747 390	5.398 883	8.828 888
55	1.000 000	1.958 089	2.874 309	3.760 017	4.584 678	8.160 431	10.781 789	12.530 382	4.584 678	8.160 431
56	1.000 000	1.956 654	2.870 378	3.741 566	4.570 828	8.098 591	10.644 257	12.301 900	3.741 566	7.473 818
57	1.000 000	1.955 122	2.865 788	3.732 410	4.555 415	8.032 123	10.497 928	12.062 024	2.866 786	6.767 145
58	1.000 000	1.953 453	2.860 797	3.722 472	4.538 936	7.960 709	10.342 480	11.810 822	1.953 453	5.038 123
59	1.000 000	1.951 639	2.855 281	3.711 705	4.521 093	7.884 098	10.177 715	11.548 581	1.000 000	5.284 081
60	1.000 000	1.949 668	2.849 508	3.700 028	4.501 790	7.802 019	10.003 456	11.275 574		4.501 790

TABLE 9

Reciprocal of Temporary Life Annuity Due—CSO 2½%

$$\frac{1000}{\ddot{a}_{x:\overline{n}|}}$$

x	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 15$	$n = 20$	To Age 60 $n = 60 - x$	To Age 65 $n = 65 - x$
20	506.781 0	342.423 8	260.282 3	211.027 5	112.751 4	80.271 03	64.263 64	41.838 68	39.847 58
21	506.801 0	342.450 7	260.313 5	211.061 6	112.767 4	80.329 05	64.335 80	42.497 06	40.390 70
22	506.821 1	342.476 6	260.345 1	211.097 6	112.847 0	80.361 76	64.414 15	43.192 04	40.961 12
23	506.843 7	342.509 3	260.361 6	211.137 4	112.901 1	80.460 18	64.499 55	43.926 73	41.560 96
24	506.868 3	842.541 9	260.420 1	211.179 8	112.959 7	80.534 46	64.592 42	44.704 19	42.192 15
25	506.893 8	342.576 0	260.462 7	211.227 1	113.024 1	80.815 89	64.593 96	45.528 33	42.857 28
26	506.921 2	342.617 1	260.508 0	211.277 5	113.093 7	80.704 13	84.804 22	46.402 56	43.558 65
27	506.951 6	342.659 6	260.558 3	211.333 6	113.169 6	80.800 56	64.924 61	47.331 78	44.296 46
28	506.986 5	342.708 1	260.614 4	211.395 4	113.253 3	80.905 90	65.055 98	48.320 84	45.080 59
29	507.024 1	342.756 9	260.674 5	211.461 7	113.343 6	81.020 26	65.198 79	46.374 91	45.907 68
30	507.064 4	342.615 1	260.736 1	211.533 1	113.441 5	61.144 61	65.354 21	50.500 33	45.783 44
31	507.107 1	342.874 7	260.806 8	211.510 5	113.546 1	81.280 01	65.523 49	51.704 28	47.712 00
32	507.154 7	342.936 6	260.885 2	211.695 4	113.664 7	81.427 87	65.708 22	52.995 10	48.698 13
33	507.204 8	343.010 9	260.968 3	211.787 8	113.791 6	81.588 99	65.909 53	54.381 84	49.746 70
34	507.262 5	343.086 7	281.060 3	211.889 7	113.930 8	81.765 07	65.129 33	65.875 46	50.863 71
35	507.322 7	343.174 2	251.156 3	211.999 5	114.061 4	81.956 51	66.368 53	57.487 65	52.054 96
36	507.390 7	343.267 7	281.268 2	212.120 2	114.246 3	82.165 57	85.829 58	59.233 27	53.828 23
37	507.463 5	343.368 0	261.366 8	212.250 5	114.425 5	82.893 10	66.918 84	51.128 52	54.691 25
38	507.541 3	343.477 2	261.513 8	212.392 6	114.620 7	82.641 14	57.223 72	63.193 84	66.153 63
39	507.629 2	343.596 0	261.654 7	212.548 6	114.834 2	82.912 02	67.561 94	65.451 51	57.728 45
40	507.722 2	343.727 6	261.805 5	212.717 3	115.066 0	83.205 86	57.930 29	57.930 29	59.421 46
41	507.825 3	343.870 0	261.972 7	212.901 7	115.319 3	83.528 69	68.332 18	70.664 12	61.253 69
42	507.936 2	344.023 6	262.153 0	213.102 0	115.595 0	83.679 44	58.770 24	73.693 81	63.239 55
43	508.057 0	344.162 0	262.350 1	213.320 6	115.695 6	84.262 24	66.248 14	77.070 58	65.899 20
44	508.190 5	344.376 5	262.565 5	213.559 5	115.224 3	84.680 06	69.769 58	60.657 94	67.756 16
45	508.334 1	344.575 6	262.799 0	213.616 6	116.581 8	85.135 44	70.337 94	65.135 44	70.337 94
46	506.490 5	344.793 2	263.053 5	214.101 6	116.672 0	85.632 46	70.658 04	90.006 63	73.176 83
47	506.662 3	345.030 3	263.331 5	214.410 3	117.397 8	85.175 00	71.634 64	95.605 05	78.320 05
48	508.846 6	345.287 9	263.633 5	214.746 0	117.861 6	88.766 78	72.372 53	102.112 36	79.611 93
49	509.051 2	345.570 7	263.664 3	215.118 1	118.368 9	87.413 37	73.178 00	109.770 80	83.718 52
50	509.271 0	345.676 7	264.322 9	215.512 0	118.621 4	88.118 61	74.056 45	118.621 38	88.118 51
51	509.511 6	346.210 4	264.714 5	215.947 3	119.524 6	88.888 87	75.015 05	130.058 22	93.115 46
52	509.772 6	348.574 2	265.141 1	216.421 7	120.182 6	89.730 00	76.060 97	143.920 32	98.842 64
53	510.056 4	346.671 6	265.606 9	216.636 5	120.901 7	90.646 02	77.202 45	151.669 80	105.478 66
54	510.369 2	347.402 7	266.113 0	217.503 0	121.685 6	61.652 41	78.447 43	165.240 65	113.265 19
55	510.707 2	347.673 4	266.685 5	218.117 6	122.542 5	92.746 99	76.806 03	218.117 83	122.542 55
56	511.078 6	346.366 4	287.267 8	216.766 4	123.476 3	93.947 37	81.288 26	267.267 77	133.800 42
57	511.477 0	348.944 4	267.623 4	216.516 0	124.500 1	65.256 89	62.904 83	346.944 41	147.772 61
58	511.914 0	349.552 9	268.636 7	220.316 0	125.617 0	96.666 61	84.668 11	511.914 03	165.514 58
59	512.389 6	350.215 0	266.418 0	221.185 5	128.837 5	66.253 88	88.590 72	1 000.000 00	189.247 67
60	512.907 8	350.937 6	270.268 2	222.133 9	128.172 0	99.965 45	88.686 49		222.133 86

TABLE 10

Single-Premium Temporary Insurance—CSO 2½%

$1000A^{1}_{x:\overline{n}|}$

x	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 15$	$n = 20$	To Age 60 $n = 60 - x$	To Age 65 $n = 65 - x$
20	2.370 63	4.753 46	7.146 46	9.556 37	11.978 81	24.547 26	38.289 54	53.899 23	151.746 86	167.887 94
21	2.446 35	4.907 18	7.383 33	9.873 52	12.382 88	25.485 85	69.833 48	56.455 33	153.483 58	190.721 16
22	2.526 64	5.071 07	7.629 95	10.218 28	12.832 15	26.518 96	41.717 51	58.225 29	155.200 61	193.465 15
23	2.614 82	5.244 47	7.904 39	10.590 68	13.308 60	27.632 42	43.656 58	62.229 15	156.897 18	186.220 16
24	2.702 65	5.436 40	6.197 13	10.990 68	13.829 45	28.836 60	45.757 04	65.482 70	158.564 38	198.976 79
25	2.808 88	5.647 49	6.516 73	11.436 63	14.404 78	30.156 31	48.046 29	69.016 29	160.202 09	201.741 90
26	2.916 86	5.868 47	8.887 87	11.919 10	15.026 09	31.582 09	50.528 15	72.842 80	161.782 88	204.494 81
27	3.034 36	5.116 09	9.254 87	12.449 07	15.702 60	33.138 55	63.225 85	76.987 75	163.335 29	207.236 69
28	3.170 67	6.385 81	8.680 17	13.025 44	16.442 02	34.835 79	56.154 61	81.478 93	164.822 11	209.960 11
29	3.616 66	6.694 00	10.134 10	13.647 50	17.236 20	36.669 95	59.325 50	66.331 70	166.232 96	212.650 25
30	3.473 59	7.011 71	10.625 22	14.316 23	16.102 63	36.665 10	62.765 06	81.579 73	167.556 84	215.298 87
31	6.638 54	7.356 64	11.153 41	15.048 33	19.040 14	40.828 96	66.491 75	97.246 76	168.788 33	217.896 73
32	3.824 30	7.730 57	11.737 80	15.844 73	20.068 14	43.185 46	70.532 99	103.374 98	168.811 38	220.435 85
33	4.019 66	6.143 28	12.369 44	16.715 49	21.186 55	45.742 54	74.912 89	109.984 26	170.909 22	222.900 68
34	4.244 18	6.593 90	13.067 02	17.688 83	22.404 00	48.522 25	79.656 11	117.119 44	171.768 50	225.281 24
35	4.477 96	8.082 93	13.820 41	18.695 14	23.727 90	51.537 83	84.786 27	124.799 12	172.463 72	227.552 92
36	4.741 67	9.520 16	14.639 82	18.822 17	25.168 02	54.613 01	90.341 47	133.068 26	172.878 38	229.706 15
37	5.024 68	10.194 86	15.532 81	21.040 09	26.733 93	58.359 88	96.339 88	141.956 18	173.285 68	231.714 62
36	5.826 98	10.826 60	16.500 76	22.357 18	28.432 81	62.208 45	102.816 79	151.488 38	173.350 38	233.550 11
39	5.668 07	11.516 03	17.562 12	23.813 51	30.282 47	66.380 16	109.816 56	161.738 68	173.179 84	235.223 31
40	6.029 21	12.252 63	18.707 77	25.877 18	32.291 20	70.899 56	117.357 70	172.702 82	172.702 92	236.668 12
41	5.429 00	13.075 31	19.955 00	27.085 94	34.475 85	75.795 34	125.486 40	184.431 12	171.902 92	237.875 98
42	6.858 71	13.956 13	21.313 81	28.936 72	86.844 89	81.092 38	134.236 34	196.948 76	170.735 66	236.806 83
43	7.326 37	14.821 40	22.792 26	30.953 43	38.423 86	86.823 00	143.646 61	210.302 58	169.163 32	238.450 05
44	7.843 80	15.872 49	24.401 00	33.148 88	42.221 22	93.025 89	153.757 45	224.506 24	167.136 02	238.706 36
45	6.399 44	17.106 70	25.147 92	35.522 45	45.259 26	99.720 62	164.599 96	239.585 37	164.599 86	239.585 37
46	8.004 51	18.350 18	28.042 53	68.109 43	46.556 14	106.850 05	176.218 29	255.563 03	161.495 92	238.023 45
47	9.668 57	18.695 77	30.110 45	40.916 09	52.135 50	114.753 47	188.646 61	272.451 77	157.758 75	237.965 75
46	10.380 75	21.162 68	32.351 38	43.864 31	56.011 43	123.158 78	201.827 01	290.251 44	153.312 85	236.346 89
48	11.170 33	22.762 09	34.793 35	47.274 46	60.225 86	132.219 16	216.087 40	306.963 21	148.081 02	234.106 22
50	12.018 17	24.494 04	37.435 35	50.864 28	64.783 17	141.858 82	231.155 93	328.567 51	141.958 62	231.155 93
51	12.946 25	26.376 53	40.812 89	54.757 70	68.722 01	152.427 90	247.165 05	348.048 12	134.848 44	227.416 87
52	13.951 18	28.426 04	43.433 10	58.877 78	75.071 63	163.661 13	264.133 50	370.358 81	126.631 16	222.786 79
53	15.054 07	30.657 35	46.621 82	63.557 33	80.861 54	175.714 78	282.077 05	382.453 17	117.172 56	217.163 97
54	15.244 03	33.072 26	50.485 00	68.509 78	87.117 78	188.605 16	300.992 42	415.259 00	106.311 88	210.409 37
55	17.541 01	35.701 57	54.479 52	73.675 64	93.882 71	202.389 29	320.888 63	436.696 88	93.882 71	202.389 29
56	18.955 50	38.555 17	58.800 18	78.682 82	101.190 00	217.100 14	341.753 28	462.667 40	78.682 92	192.836 42
57	20.487 73	41.650 05	63.478 86	65.960 53	109.068 42	232.768 72	363.550 94	487.051 91	63.476 86	181.866 03
58	22.156 67	45.011 24	68.549 14	82.743 85	117.580 39	249.427 40	386.250 00	511.722 58	45.011 24	168.960 95
58	23.970 32	48.657 35	74.033 20	100.082 23	126.741 24	267.094 88	409.785 07	536.531 74	23.970 82	153.971 16
50	25.941 57	52.606 99	79.979 80	107.993 56	136.507 24	285.795 14	434.112 98	561.320 76		136.607 24

TABLE 11

Foreborne Life Annuity Due—CSO 2½%

$_nu_x$

x	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 15$	$n = 20$	To Age 60 $n = 60 - x$	To Age 65 $n = 65 - x$
20	1.027 496 7	2.083 412 5	3.168 704 2	4.284 404 9	5.431 561 5	11.683 141	18.936 794	27.465 329	90.093 747	125.526 041
21	1.027 578 8	2.083 664 5	3.169 250 4	4.285 352 9	5.433 134 8	11.891 624	18.964 180	27.538 000	86.324 344	120.524 132
22	1.027 661 4	2.083 938 4	3.169 817 9	4.285 408 6	5.434 818 3	11.700 810	18.994 145	27.617 470	82.655 812	115.656 078
23	1.027 754 8	2.084 222 2	3.170 459 0	4.287 539 7	5.436 637 4	11.710 824	19.026 845	27.704 242	79.085 739	110.918 875
24	1.027 847 3	2.084 547 2	3.171 142 3	4.288 759 9	5.438 654 2	11.721 572	19.062 424	27.799 022	75.811 762	106.308 789
25	1.027 960 6	2.084 893 2	3.171 880 9	4.290 120 5	5.440 870 1	11.733 579	19.101 397	27.902 838	72.231 599	101.823 393
26	1.028 073 8	2.085 261 3	3.172 703 9	4.291 511 2	5.443 281 9	11.746 502	19.143 866	28.016 339	68.943 015	97.459 519
27	1.023 197 9	2.085 681 2	8.173 612 2	4.293 238 3	5.445 885 3	11.760 660	19.190 308	28.140 507	65.743 880	93.214 343
28	1.028 342 0	2.086 142 8	3.174 599 1	4.294 986 7	5.448 719 8	11.776 100	19.240 981	28.278 561	82.632 105	89.085 091
29	1.028 498 4	2.086 638 2	3.175 651 0	4.296 885 5	5.451 770 1	11.792 870	19.296 284	28.425 382	59.605 889	85.069 083
30	1.028 662 5	2.087 165 3	3.178 797 1	4.298 929 4	5.455 134 0	11.811 250	19.356 844	28.588 711	56.662 645	81.163 759
31	1.028 838 1	2.087 742 8	3.178 030 4	4.301 192 5	5.468 777 7	11.831 265	19.423 015	28.757 735	53.801 152	77.366 639
32	1.029 033 7	2.088 364 0	3.179 404 4	4.303 642 8	5.462 785 2	11.853 152	19.495 354	28.964 194	51.019 412	73.675 322
33	1.029 240 7	2.089 059 7	3.180 886 1	4.306 338 8	5.467 155 0	11.877 012	19.574 576	29.179 805	48.315 633	70.087 472
34	1.029 478 5	2.089 807 5	3.182 521 0	4.309 278 3	5.471 887 4	11.903 031	19.661 144	29.418 906	45.688 140	66.600 851
35	1.029 726 4	2.090 630 8	3.184 298 8	4.312 450 7	5.477 082 8	11.931 512	19.756 086	29.877 509	43.135 294	63.213 285
36	1.030 006 2	2.091 528 5	3.186 213 8	4.315 943 8	5.482 725 3	11.962 598	19.860 039	29.964 298	40.655 548	59.922 719
37	1.030 306 4	2.092 489 3	3.188 325 0	4.319 728 8	5.488 892 2	11.998 538	19.973 986	30.280 312	38.247 387	56.727 147
38	1.030 827 4	2.093 556 1	3.190 612 8	4.823 871 2	5.495 603 0	12.033 879	20.098 888	30.628 694	35.909 381	53.824 668
39	1.030 989 8	2.094 709 7	3.193 119 2	4.328 876 4	5.502 910 1	12.074 203	20.236 091	31.013 292	33.640 147	50.813 449
40	1.031 373 8	2.095 970 0	3.195 836 5	4.333 273 3	5.510 912 4	12.118 816	20.386 873	31.438 349	31.438 349	47.691 714
41	1.031 799 3	2.097 338 9	3.198 793 8	4.338 646 0	5.519 632 9	12.157 175	20.552 140	31.908 759	29.302 732	44.857 802
42	1.032 256 9	2.098 824 5	3.202 039 3	4.344 493 8	5.529 137 8	12.220 330	20.734 185	32.429 928	27.232 081	42.110 096
43	1.032 755 5	2.100 461 9	3.205 571 8	4.350 867 6	5.539 537 7	12.278 495	20.934 511	33.008 927	25.225 245	39.447 072
44	1.033 307 7	2.102 239 5	3.209 417 1	4.357 839 9	5.550 848 6	12.342 303	21.155 233	33.652 323	23.281 120	36.887 265
45	1.033 901 3	2.104 189 4	3.218 621 6	4.365 411 4	5.563 243 4	12.412 187	21.398 657	34.369 281	21.398 657	34.359 281
46	1.034 548 5	2.108 286 1	3.218 185 7	4.373 717 4	5.578 764 1	12.488 824	31.667 435	35.169 723	19.576 873	31.951 815
47	1.035 259 7	2.108 577 1	3.223 195 7	4.382 789 7	5.591 538 2	12.572 966	21.964 451	36.065 389	17.814 825	29.813 620
48	1.038 023 8	2.111 095 3	3.228 648 5	4.392 654 8	5.607 860 8	12.665 322	22.293 571	37.069 909	16.111 820	27.353 507
49	1.036 871 7	2.113 835 2	3.234 604 0	4.403 438 2	5.625 325 1	12.788 820	22.658 164	38.199 273	14.466 424	25.170 371
50	1.037 785 1	2.118 819 8	3.241 087 8	4.415 243 9	5.644 608 4	12.878 433	23.063 145	39.473 160	12.878 433	23.063 145
51	1.038 784 6	2.120 071 2	3.248 193 9	4.428 127 6	5.665 697 6	13.001 291	23.513 695	40.914 951	11.346 912	21.030 853
52	1.039 870 1	2.123 635 4	3.255 938 0	4.442 205 4	5.688 790 1	13.138 580	24.015 915	42 551 608	9.871 152	19.072 555
53	1.041 064 0	2.127 514 0	3.264 394 2	4.457 512 7	5.714 047 5	13.285 991	24.575 807	44.417 084	8.450 493	17.187 373
54	1.042 355 3	2.131 742 0	3.273 640 3	4.474 444 4	5.741 702 8	13.450 792	25.204 482	46.552 364	7.084 303	15.374 479
55	1.043 766 5	2.138 368 9	3.283 736 4	4.492 865 3	5.772 002 4	13.633 080	25.908 985	49.007 282	5.772 002	13.633 080
56	1.045 309 7	2.141 411 9	3.294 773 4	4.513 025 7	5.805 212 8	13.834 918	28.702 052	51.844 133	4.513 026	11.962 447
57	1.045 986 6	2.148 919 1	3.306 839 8	4.535 099 5	5.841 602 2	14.058 742	27.598 929	55.140 078	3.306 840	10.381 867
58	1.048 819 3	2.152 936 4	3.320 040 2	4.559 259 7	5.881 830 8	14.307 298	28.510 345	58.992 579	2.152 936	8.830 665
59	1.050 818 1	2.159 510 2	3.334 468 2	4.585 813 8	5.925 508 1	14.583 702	29.762 188	63.525 102	1.050 818	7.368 180
60	1.052 999 4	2.166 683 0	3.350 313 0	4.514 871 9	5.973 759 0	14.891 857	31.076 256	68.895 771		5.973 769

TABLE 12

Accumulated Cost of Insurance—CSO 2½%

$1000\,{}_nk_x$

x	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 15$	$n = 20$	To Age 60 $n = 60 - x$	To Age 65 $n = 65 - x$
20	2.435 81	5.018 86	7.754 22	10.656 82	13.731 49	32.335 87	58.203 08	95.133 24	571.995 2	940.298 7
21	2.515 87	5.182 00	8.013 21	11.014 26	14.210 87	33.623 54	60.833 57	100.020 83	563.059 4	928.441 1
22	2.596 53	5.355 99	8.283 04	11.403 08	14.722 05	35.015 71	63.701 54	105.359 22	554.077 6	916.522 4
23	2.687 39	5.540 12	8.583 48	11.823 29	15.276 62	36.534 64	66.834 02	111.197 97	545.057 3	904.552 7
24	2.777 90	5.744 02	8.904 12	12.275 13	15.883 57	38.181 64	70.245 47	117.581 16	535.973 5	892.498 7
25	2.888 44	5.968 38	9.256 81	12.779 49	18.554 83	39.992 68	73.988 45	124.588 08	526.838 1	880.378 2
25	2.998 85	6.203 35	9.639 69	13.325 56	17.280 65	41.955 43	78.065 61	132.251 78	517.597 6	868.114 3
27	8.119 94	6.468 89	10.064 41	13.926 02	18.072 12	44.107 18	82.531 39	140.657 52	508.265 8	855.731 3
28	8.260 53	6.764 62	10.531 65	14.579 86	16.928 50	46.459 87	87.418 71	149.887 24	498.823 5	843.201 8
29	3.411 17	7.082 09	11.080 82	15.286 42	19.870 62	49.014 76	92.748 95	159.998 68	489.227 7	830.468 2
30	8.578 15	7.420 69	11.571 47	18.047 04	20.889 38	51.806 86	98.585 82	171.106 89	479.486 7	817.515 5
21	3.744 50	7.768 64	12.153 50	16.861 06	21.993 95	54.850 37	104.970 75	183.310 25	489.527 1	804.325 9
32	3.935 33	8.187 62	12.798 27	17.789 79	23.207 74	58.183 10	111.968 86	196.741 75	459.402 8	790.891 2
33	4.137 22	8.628 47	13.496 04	18.785 15	24.531 42	61.821 24	119.641 13	211.524 75	449.062 8	777.170 2
34	4.869 29	9.110 25	14.267 76	19.877 12	25.976 04	65.802 01	128.054 77	227.834 84	438.501 1	763.155 1
35	4.611 07	9.633 58	15.102 50	21.055 18	27.551 39	70.151 46	137.284 07	245.810 92	427.666 4	748.777 7
35	4.884 15	10.209 13	16.011 93	22.351 88	29.271 49	74.912 00	147.420 27	265.871 87	416.552 1	734.042 7
37	5.176 96	10.825 64	17.004 84	23.756 72	31.145 69	80.111 17	158.548 33	287.827 61	405.143 0	718.889 7
38	5.490 13	11.508 98	16.083 27	25.291 74	33.187 62	85.804 72	170.778 19	211.933 42	393.395 3	703.300 8
39	5.843 72	12.245 42	19.268 25	26.989 73	35.419 48	92.038 19	184.251 86	338.896 56	381.307 1	687.260 1
40	8.218 37	13.049 54	20.550 42	28.789 90	37.853 89	98.865 27	199.075 20	368.827 16	368.827 2	670.699 5
41	6.633 43	13.927 35	21.949 69	30.786 05	40.513 93	106.349 17	215.421 70	402.132 73	355.951 1	653.613 2
42	7.079 95	14.878 20	23.478 81	32.958 97	43.413 24	114.551 80	233.460 07	439.240 13	342.638 9	635.948 3
43	7.566 34	15.923 42	25.147 44	35.331 84	46.587 11	123.551 48	253.391 01	480.711 08	328.874 5	817.683 4
44	8.105 06	17.064 02	26.989 15	37.929 58	50.050 62	133.443 49	275.445 17	527.120 07	314.631 2	598.782 8
45	8.684 19	18.299 81	28.964 54	40.762 29	53.837 09	144.299 27	299.865 58	579.189 07	299.865 6	579.189 1
45	9.315 59	19.653 54	31.118 24	43.845 74	57.975 77	156.237 22	325.951 74	637.776 55	284.563 6	558.883 7
47	10.009 48	21.124 76	33.485 88	47.224 44	62.504 39	169.380 47	357.068 13	703.887 63	268.597 2	537.829 4
48	10.754 71	22.733 44	36.065 74	50.912 90	67.450 22	183.847 73	390.595 69	778.699 00	252.229 7	515.977 4
49	11.582 19	24.493 14	38.891 48	54.949 48	72.878 17	199.808 93	427.986 48	863.660 28	235.151 3	493.314 8
50	12.473 31	28.405 45	41.965 65	59.381 17	78.807 48	217.412 92	489.776 37	960.482 53	217.412 9	489.776 4
51	13.448 38	28.491 97	45.334 20	64.186 40	85.304 32	238.886 93	516.601 06	1 071.311 47	199.005 2	445.349 9
52	14.507 41	30.775 39	49.010 94	69.464 69	92.426 48	258.386 75	569.193 93	1 198.672 36	179.899 8	419.997 3
53	15.872 24	33.268 10	53.032 84	75.250 10	100.238 22	282.250 47	628.429 13	1 345.764 19	160.079 9	393.696 8
54	16.932 05	35.981 82	57.428 50	81.575 15	108.795 94	308.703 55	695.307 95	1 516.489 91	139.513 2	366.405 8
55	18.308 71	38.952 63	82.233 26	88.509 79	118.196 13	338.118 04	771.105 81	1 715.776 92	118.196 1	338.118 0
56	19.814 36	42.195 74	87.494 05	96.112 42	128.522 77	370.874 68	857.318 10	1 949.828 12	96.112 4	308.813 5
57	21.450 37	45.735 91	73.248 57	104.447 15	139.864 36	407.418 52	955.701 86	2 226.498 77	73.248 5	278.473 7
58	23.238 34	49.607 72	79.553 35	113.592 08	152.362 87	448.280 68	1 068.481 29	2 555.947 12	49.607 7	247.102 8
59	25.168 45	53.839 87	86.454 73	123.651 68	166.111 66	494.061 42	1 198.343 36	2 951.291 86	25.186 4	214.699 0
60	27.315 46	58.462 50	94.036 33	134.695 28	181.274 57	545.502 45	1 348.594 48	3 429.739 45		181.274 5

TABLE 13

Net Level Annual Premiums per $1000—CSO 2½%

x	$1000P_x$	$1000\,{}_{20}P_x$
20	12.490 76	21.764 60
21	12.878 29	22.231 49
22	13.282 05	22.710 37
23	13.703 09	23.202 04
24	14.142 20	23.706 75
25	14.600 66	24.225 51
26	15.079 05	24.758 14
27	15.578 60	25.305 77
28	16.101 07	25.869 02
29	16.648 73	26.448 02
30	17.217 17	27.043 60
31	17.813 83	27.656 87
32	18.438 40	28.288 41
33	19.092 31	28.939 54
34	19.777 50	29.611 49
35	20.495 27	30.304 67
36	21.248 00	31.021 03
37	22.037 34	31.761 35
38	22.865 65	32.527 45
39	23.735 57	33.321 43
40	24.649 04	34.144 39
41	25.609 23	34.999 08
42	26.618 71	35.887 33
43	27.680 82	36.812 10
44	28.798 92	37.776 28
45	29.978 12	38.782 41
46	31.215 61	39.834 47
47	32.524 48	40.936 32
48	33.903 82	42.091 86
49	35.359 96	43.306 54
50	36.897 14	44.584 57
51	38.521 39	45.932 43
52	40.238 39	47.356 27
53	42.054 68	48.863 39
54	43.978 42	50.460 81
55	46.011 53	52.157 74
56	48.167 70	53.963 33
57	50.453 10	55.887 48
58	52.877 24	57.941 79
59	55.449 72	60.138 20
60	58.181 10	62.489 93

TABLE 14

1937 Standard Annuity Table—2½% Interest

x	l_x	d_x	D_x	N_x	S_x	a_x
5	1 000 000	1 234	883 854	28 254 266	709 323 963	30.967 12
6	998 766	1 241	861 233	27 370 412	681 069 697	30.780 50
7	997 525	1 247	839 183	26 509 179	653 699 285	30.589 27
8	996 278	1 250	817 692	25 669 996	627 190 106	30.393 23
9	995 028	1 250	796 747	24 852 304	601 520 110	30.192 22
10	993 778	1 249	776 338	24 055 557	576 667 806	29.985 93
11	992 529	1 247	756 451	23 279 219	552 612 249	29.774 26
12	991 282	1 246	737 074	22 522 768	529 333 030	29.557 00
13	990 036	1 244	718 192	21 785 694	506 810 262	29.334 08
14	988 792	1 245	699 795	21 067 502	485 024 568	29.105 25
15	987 547	1 246	681 867	20 367 707	463 957 066	28.870 50
16	986 301	1 250	664 397	19 685 840	443 589 359	28.629 63
17	985 051	1 258	647 371	19 021 443	423 903 519	28.382 60
18	983 793	1 269	630 775	18 374 072	404 882 076	28.129 36
19	982 524	1 285	614 596	17 743 297	386 508 004	27.869 85
20	981 239	1 306	598 822	17 128 701	368 764 707	27.603 99
21	979 933	1 333	583 439	16 529 879	351 636 006	27.331 80
22	978 600	1 368	568 434	15 946 440	335 106 127	27.053 28
23	977 232	1 409	553 795	15 378 006	319 159 687	26.768 41
24	975 823	1 460	539 508	14 824 211	303 781 681	26.477 28
25	974 363	1 521	525 562	14 284 703	288 957 470	26.179 86
26	972 842	1 590	511 943	13 759 141	274 672 767	25.876 31
27	971 252	1 672	498 641	13 247 198	260 913 626	25.566 60
28	969 580	1 767	485 641	12 748 557	247 666 428	25.250 99
29	967 813	1 874	472 933	12 262 916	234 917 871	24.929 50
30	965 939	1 995	460 504	11 789 983	222 654 955	24.602 35
31	963 944	2 132	448 345	11 329 479	210 864 972	24.269 56
32	961 812	2 286	436 442	10 881 134	199 535 493	23.931 45
33	959 526	2 458	424 785	10 444 692	188 654 359	23.588 18
34	957 068	2 644	413 363	10 019 907	178 209 667	23.239 97
35	954 424	2 845	402 167	9 606 544	168 189 760	22.886 95
36	951 579	3 060	391 188	9 204 377	158 583 216	22.529 29
37	948 519	3 291	380 420	8 813 189	149 378 839	22.167 00
38	945 228	3 537	369 853	8 432 769	140 565 650	21.800 33
39	941 691	3 802	359 482	8 062 916	132 132 881	21.429 26

TABLE 14—*Continued*

x	l_x	d_x	D_x	N_x	S_x	a_x
40	937 889	4 085	349 299	7 703 434	124 069 965	21.053 98
41	933 804	4 388	339 295	7 354 135	116 366 531	20.674 75
42	929 416	4 710	329 464	7 014 840	109 012 396	20.291 67
43	924 706	5 056	319 799	6 685 376	101 997 556	19.904 93
44	919 650	5 424	310 293	6 365 577	95 312 180	19.514 73
45	914 226	5 816	300 940	6 055 284	88 946 603	19.121 23
46	908 410	6 234	291 732	5 754 344	82 891 319	18.724 76
47	902 176	6 679	282 663	5 462 612	77 136 975	18.325 53
48	895 497	7 149	273 728	5 179 949	71 674 363	17.923 71
49	888 348	7 651	264 919	4 906 221	66 494 414	17.519 70
50	880 697	8 180	256 232	4 641 302	61 588 193	17.113 67
51	872 517	8 741	247 661	4 385 070	56 946 891	16.705 94
52	863 776	9 333	239 199	4 137 409	52 561 821	16.296 93
53	854 443	9 957	230 844	3 898 210	48 424 412	15.886 77
54	844 486	10 612	222 589	3 667 366	44 526 202	15.475 95
55	833 874	11 302	214 431	3 444 777	40 858 836	15.064 73
56	822 572	12 021	206 366	3 230 346	37 414 059	14.653 48
57	810 551	12 774	198 390	3 023 980	34 183 713	14.242 60
58	797 777	13 556	190 501	2 825 590	31 159 733	13.832 42
59	784 221	14 368	182 697	2 635 089	28 334 143	13.423 27
60	769 853	15 207	174 975	2 452 392	25 699 054	13.015 67
61	754 646	16 072	167 335	2 277 417	23 246 662	12.609 93
62	738 574	16 956	159 777	2 110 082	20 969 245	12.206 42
63	721 618	17 859	152 301	1 950 305	18 859 163	11.805 60
64	703 759	18 773	144 909	1 798 004	16 908 858	11.407 81
65	684 986	19 694	137 604	1 653 095	15 110 854	11.013 42
66	665 292	20 615	130 388	1 515 491	13 457 759	10.622 93
67	644 677	21 526	123 266	1 385 103	11 942 268	10.236 70
68	623 151	22 420	116 244	1 261 837	10 557 165	9.855 07
69	600 731	23 286	109 328	1 145 593	9 295 328	9.478 50
70	577 445	24 113	102 527	1 036 265	8 149 735	9.107 24
71	553 332	24 889	95 849.8	933 737.9	7 113 469.9	8.741 68
72	528 443	25 600	89 305.8	837 888.1	6 179 732.0	8.382 24
73	502 843	26 232	82 906.8	748 582.3	5 341 843.9	8.029 20
74	476 611	26 770	76 665.2	665 675.5	4 593 261.6	7.682 89

TABLE 14—*Continued*

x	l_x	d_x	D_x	N_x	S_x	a_x
75	449 841	27 199	70 594.2	589 010.3	3 927 586.1	7.343 61
76	422 642	27 506	64 708.1	518 416.1	3 338 575.8	7.011 61
77	395 136	27 672	59 021.3	453 708.0	2 820 159.7	6.687 19
78	367 464	27 688	53 549.2	394 686.7	2 366 451.7	6.370 54
79	339 776	27 539	48 306.7	341 137.5	1 971 765.0	6.061 91
80	312 237	27 215	43 308.7	292 830.8	1 630 627.5	5.761 48
81	285 022	26 709	38 569.6	249 522.1	1 337 796.7	5.469 40
82	258 313	26 018	34 102.7	210 952.5	1 088 274.6	5.185 80
83	232 295	25 141	29 919.8	176 849.8	877 322.1	4.910 79
84	207 154	24 083	26 030.9	146 930.0	700 472.3	4.644 45
85	183 071	22 854	22 443.5	120 899.1	553 542.3	4.386 82
86	160 217	21 469	19 162.7	98 455.6	432 643.2	4.137 88
87	138 748	19 950	16 190.1	79 292.9	334 187.6	3.897 62
88	118 798	18 320	13 524.1	63 102.8	254 894.7	3.665 95
89	100 478	16 611	11 159.6	49 578.7	191 791.9	3.442 70
90	83 867	14 856	9 087.48	38 419.13	142 213.25	3.227 70
91	69 011	13 092	7 295.36	29 331.65	103 794.12	3.020 59
92	55 919	11 355	5 767.18	22 036.29	74 462.47	2.820 98
93	44 564	9 680	4 483.99	16 269.11	52 426.18	2.628 27
94	34 884	8 100	3 424.39	11 785.12	36 157.07	2.441 52
95	26 784	6 644	2 565.12	8 360.73	24 371.95	2.259 39
96	20 140	5 333	1 881.78	5 795.61	16 011.22	2.079 86
97	14 807	4 198	1 349.75	3 913.83	10 215.61	1.899 67
98	10 609	3 244	943.486	2 564.078	6 301.78	1.717 66
99	7 365	2 444	639.014	1 620.592	3 737.70	1.536 08
100	4 921	1 782	416.550	981.578	2 117.11	1.356 45
101	3 139	1 248	259.227	565.028	1 135.53	1.179 66
102	1 891	830	152.355	305.801	570.50	1.007 16
103	1 061	517	83.398 3	153.446	264.70	0.839 92
104	544	295	41.717 4	70.047 7	111.25	0.679 10
105	249	152	18.629 2	28.330 3	41.20	0.520 75
106	97	67	7.080 14	9.701 1	12.87	0.370 18
107	30	24	2.136 33	2.620 9	3.17	0.226 84
108	6	5	.416 84	.484 6	.55	0.162 65
109	1	1	.067 78	.067 8	.07	

APPENDIX FOUR

Answers to Problems

SET 1

1. 5/36. **2.** 1/36, 1/36. **3.** 1/3, 3/5. **4.** 2/91. **5.** 1/5525, 16/5525. **6.** 11/20. **7.** 5/16, 1/2, 13/16. **8.** 44/125. **9.** 5/72. **10.** 7/162. **11.** (*a*) 0.72; (*b*) 0.28; (*c*) 0.18. **12.** (*a*) 1/5; (*b*) 1/120; (*c*) 119/120; (*d*) 4/5; (*e*) 7/24. **13.** 17/47. **14.** 2/3, 1/3. **15.** 0.0305. **16.** 4/9. **17.** 3439/40,000. **18.** 195/256. **19.** 0.504. **20.** 5/14.

SET 2

1. 0.455. **2.** 0.133. **3.** 0.291.

SET 3

1. (0.99645)(0.99627).
2. (0.99594)(0.99579)(0.99561)(0.99559)(0.99557)(0.00446). **3.** 0.00435.
4. 1 − (0.99608)(0.99598)(0.99727)(0.99641). **5.** $d_{99} = 100$.

SET 4

1. (*a*) $\dfrac{677{,}771}{939{,}197}$; (*b*) $\dfrac{181{,}765}{951{,}483}$; (*c*) $\dfrac{5785}{933{,}692}$; (*d*) $\dfrac{134{,}217}{577{,}882}$; (*e*) $\dfrac{(506{,}403)(480{,}850)}{(955{,}942)(953{,}743)}$.
3. 0.297. **4.** 45.16, 0.5807 (by interpolation between ${}_{45}p_{21}$ and ${}_{46}p_{21}$).
5. $q_{98} = 0.9000$, $\mathring{e}_{98} = 0.60$. **6.** $1/(100 - x)$. **7.** 0.315. **8.** 0.315.
9. 0.456. **10.** 0.584.

SET 6

1. 49/54. **2.** 47/85. **3.** \$3.00, \$3.90, \$3.90. **4.** \$27 annually. **5.** 33/66,640. **6.** 4/19. **7.** (*a*) 0.0269; (*b*) 0.0023; (*c*) 0.1185. **10.** 13/15. **11.** 0.2736.

SET 7

1. \$5657.04, \$6400.43, \$13,425.32. **2.** \$781.20, \$610.27, \$290.94. **3.** \$1309.16, \$1277.23. **4.** \$435.65. **5.** \$1946.40. **6.** \$858.42. **7.** \$5187.80. **8.** \$1317.38. **9.** \$11,558.92. **11.** \$12,428.83. **12.** \$12,952.55. **13.** \$1218.80. **14.** \$892.59.

SET 8

1. (*a*) \$11,092.77; (*b*) \$54,876.00. **2.** (*a*) \$1651.43; (*b*) \$1800.06. **3.** \$2711.42. **4.** \$1329.65. **5.** \$1420.45. **6.** \$102,178.84. **14.** 723. **15.** (*a*) \$14,687.22, \$18,214.48; (*b*) \$57,741.00, \$59,971.86. **16.** (*a*) \$1426.97; (*b*) \$1536.61. **17.** (*a*) \$1244.96; (*b*) \$1327.60. **18.** \$2860.40, \$2987.05.

SET 9

4. (*a*) $816.04; (*b*) $775.90; (*c*) $716.04; (*d*) $167.38; (*e*) $544.52; (*f*) $275.91; (*g*) $247.30; (*h*) $961.95. **5.** $3301.25. **6.** $3348.82. **7.** $475.37. **8.** $423.62. **9.** $3034.46. **10.** $1481.13. **11.** $8152.46. **13.** $a_{95} = 1.409$, $a_{96} = 0.957$, $a_{97} = 0.702$, $a_{98} = 0.300$, $\mathring{e}_{95} = 2.200$. **14.** (*a*) $4308.72; (*b*) $5922.36; (*c*) $6270.45; (*d*) $6623.78. **15.** (*a*) $4308.72; (*b*) $7709.22; (*c*) $8077.69; (*d*) $8448.58.

SET 10

1. (*a*) $566.57; (*b*) $535.22; (*c*) $342.09; (*d*) $28.97. **2.** (*a*) $1531.20; (*b*) $2032.29; (*c*) $2747.71; (*d*) $3195.57; (*e*) $3905.32; (*f*) $5001.91; (*g*) $5139.13. **3.** $3037.90. **4.** (*a*) $16,650.53; (*b*) $16,315.76; (*c*) $12,775.05. **6.** 0.77628. **7.** 0.51607. **8.** $194.23 (by interpolating between ${}_{44}E_{22}$ and ${}_{45}E_{22}$). **9.** (*a*) $413.66; (*b*) $457.25; (*c*) $467.01; (*d*) $476.24. **10.** (*a*) $413.66; (*b*) $500.93; (*c*) $508.22; (*d*) $515.07.

SET 11

1. $8152.46. **2.** $2142.38. **3.** $(10{,}000D_{40})/N_{55}$. **4.** $119,054.41. **5.** $47,694.54. **6.** $88,496.81. **7.** $273.09. **8.** $273.09.

SET 12

1. $42,612.56. **2.** $13,630.74. **3.** $608.16. **4.** $4485.40. **5.** $15,589.16. **7.** None. **8.** $2181.52. **9.** $3356.08.

SET 13

2. (*a*) $43.87; (*b*) $53.90; (*c*) $91.58; (*d*) $172.70; (*e*) $328.57; (*f*) $561.32; (*g*) $772.53. 3. (*a*) $3.64; (*b*) $11.15; (*c*) $40.83; (*d*) $97.25; (*e*) $178.11; (*f*) $217.90. **4.** $97.02. **5.** $3657.88. **6.** $3177.49. **7.** (*a*) $153.97; (*b*) $177.04. **8.** $1537.84. **9.** $826.73. **11.** (*a*) $1251.65; (*b*) $1225.75; (*c*) $1184.94. **12.** (*a*) $1078.64; (*b*) $1057.21; (*c*) $1036.62. **13.** (*a*) $979.55; (*b*) $962.01; (*c*) $945.16.

SET 14

1. (*a*) $331.95; (*b*) $338.68; (*c*) $571.46. **2.** (*a*) $3013; (*b*) $2953; (*c*) $1750. **3.** $4929.81. **4.** $6709.55. **5.** $19,640. **6.** (*a*) $840.33; (*b*) $834.81.

SET 15

2. (*a*) $618.80; (*b*) $622.99; (*c*) $623.63; (*d*) $647.78; (*e*) $705.80; (*f*) $801.50; (*g*) $875.55. **4.** (*a*) $4644.02; (*b*) $4213.38; (*c*) $3921.02; (*d*) $2535.47; (*e*) $2154.48; (*f*) $1881.34. **5.** $12,580. **6.** $12,837. **7.** $1823.39. **8.** $393.35. **9.** $2220. **10.** $592.96. **11.** $651.90. **13.** $660.59. **14.** $731.24.

SET 16

1. (*a*) $17.22; (*b*) $18.37; (*c*) $23.27; (*d*) $33.58; (*e*) $11.87; (*f*) $22.39; (*g*) $187.14; (*h*) $31.28; (*i*) $3.47. **2.** (*a*) $1000\dfrac{M_{45}}{N_{45}}$; (*b*) $1000\dfrac{M_{45}}{N_{45}-N_{75}}$; (*c*) $1000\dfrac{M_{45}-M_{55}}{N_{45}-N_{55}}$; (*d*) $1000\dfrac{M_{45}-M_{85}+D_{85}}{N_{45}-N_{85}}$; (*e*) $1000\dfrac{M_{45}-M_{65}+D_{65}}{N_{45}-N_{65}}$.
3. (*a*) $\dfrac{M_{50}}{N_{50}}$; (*b*) $\dfrac{M_{15}}{N_{15}-N_{85}}$; (*c*) $\dfrac{M_{40}-M_{50}}{N_{40}-N_{50}}$; (*d*) $\dfrac{M_{40}-M_{50}+D_{50}}{N_{40}-N_{50}}$;

(e) $\dfrac{M_{20} - M_{60} + D_{60}}{N_{20} - N_{30}}$. **4.** \$77.35. **5.** \$2487. **6.** $\dfrac{1000 d_{\overline{10|}} M_x}{N_x}$. **7.** \$492.04.
8. \$10,400. **10.** 0.882.

SET 17

4. $0.51(100 - x)$, $\dfrac{x}{51(100 - x)}$.

SET 18

1. \$35.42. **5.** \$80.68. **6.** \$74.86. **7.** \$34.34.

SET 19

1. \$609.99. **2.** \$40.20. **3.** \$274.51. **4.** \$243.60. **6.** \$14.48. **8.** \$694.
9. \$7.01.

SET 20

1. (a) \$178.32; (b) \$353.86; (c) \$234.56; (d) \$83.63; (e) \$449.30. **2.** \$360.04.

3. (a) $1000 \dfrac{M_{30}}{N_{30}} \cdot \dfrac{N_{30} - N_{40}}{D_{40}} - 1000 \dfrac{M_{30} - M_{40}}{D_{40}}$;

(b) $1000 \dfrac{M_{30}}{N_{30} - N_{45}} \cdot \dfrac{N_{30} - N_{40}}{D_{40}} - 1000 \dfrac{M_{30} - M_{40}}{D_{40}}$;

(c) $1000 \dfrac{M_{30} - M_{50}}{N_{30} - N_{50}} \cdot \dfrac{N_{30} - N_{40}}{D_{40}} - 1000 \dfrac{M_{30} - M_{40}}{D_{40}}$;

(d) $1000 \dfrac{M_{30} - M_{70} + D_{70}}{N_{30} - N_{70}} \cdot \dfrac{N_{30} - N_{40}}{D_{40}} - 1000 \dfrac{M_{30} - M_{40}}{D_{40}}$;

(e) $1000 \dfrac{M_{30} - M_{65} + D_{65}}{N_{30} - N_{50}} \cdot \dfrac{N_{30} - N_{40}}{D_{40}} - 1000 \dfrac{M_{30} - M_{40}}{D_{40}}$;

(f) 0; (g) 1000; (h) $1000 \dfrac{M_{40}}{D_{40}}$.

SET 21

1. (a) \$143.17; (b) \$258.17; (c) \$255.62; (d) \$305.14; (e) \$614.04.

2. (a) $1000 \dfrac{M_{52}}{D_{52}} - 1000 \dfrac{M_{40}}{N_{40}} \cdot \dfrac{N_{52}}{D_{52}}$;

(b) $1000 \dfrac{M_{52}}{D_{52}} - 1000 \dfrac{M_{40}}{N_{40} - N_{55}} \cdot \dfrac{N_{52} - N_{55}}{D_{52}}$;

(c) $1000 \dfrac{M_{52} - M_{60} + D_{60}}{D_{52}} - 1000 \dfrac{M_{40} - M_{60} + D_{60}}{N_{40} - N_{60}} \cdot \dfrac{N_{52} - N_{60}}{D_{52}}$;

(d) $1000 \dfrac{M_{52}}{D_{52}}$;

(e) $1000 \dfrac{M_{52} - M_{55}}{D_{52}} - 1000 \dfrac{M_{40} - M_{55}}{N_{40} - N_{55}} \cdot \dfrac{N_{52} - N_{55}}{D_{52}}$;

(f) $1000 \dfrac{M_{52} - M_{70} + D_{70}}{D_{52}}$.

7. $\frac{1}{3}$, $\frac{1}{2}$.

SET 22

1. $P = \$39.1088$, ${}_1V = \$35.12$, ${}_5V = \$183.52$. **2.** $P = \$224.8596$, ${}_1V = \$228.26$, ${}_4V = \$951.90$, ${}_6V = \$1000.00$. **3.** $P = \$15.8257$, ${}_1V = \$10.57$, ${}_5V = \$61.63$. **5.** \$389.07.

SET 23

1. (*a*) $P = \$17.22$, ${}_{11}(MV) = \$168.25$ (prospective); (*b*) $P = \$40.96$, ${}_{11}(MV) = \$477.34$ (prospective); (*c*) \$507.41; (*d*) \$595.68. *Note:* The word "prospective" after the answers to (*a*) and (*b*) implies that the required mean reserve was calculated in three steps as follows: First, calculate the net premium to the nearest cent. Second, using this value of the net premium, calculate the two necessary terminal reserves to the nearest cent by the prospective method. Third, using these values of the net premium and the terminal reserves, calculate the mean reserve to the nearest cent—raising to the next higher cent if the answer doesn't come out even.

If the retrospective method is used, the answer may differ by a few cents because of dropping decimals, especially in the net premium. As pointed out in the text, P is normally calculated to at least 4 places in a practical actuarial problem. The reason for dropping places in our calculations is that many readers of this book will not have ready access to a calculating machine. It is felt that the saving in arithmetic more than compensates for loss of accuracy *provided* the reader is fully aware that the dropping of places is purely a textbook device.

2. \$563.32. **3.** $P = \$10.83$, ${}_{10}(MV) = \$125.57$ (prospective). **4.** $P = \$41.73$, ${}_{1}(MV) = \$40.00$ (retrospective), cost = \$4.51.

SET 24

1. $P = \$64.39$, ${}_{10}V = \$584.76$ (prospective). **2.** $P = \$22.7104$, ${}_{1}V = \$20.74$, ${}_{5}V = \$108.70$. **3.** ${}_{3}I = \$64.68$, ${}_{5}I = \$108.65$. **4.** \$21.73, \$42.71, \$64.19, \$86.18, \$108.68. **5.** \$5370.46, \$5483.38, \$5598.45, \$5715.69, \$5835.12. **6.** $P = \$19.98$, ${}_{3}V = \$35.68$, ${}_{20}V = \$183.87$, ${}_{30}V = \$369.76$ (prospective). **7.** \$7489.37.

8. $\dfrac{(N_{20} - N_{30})M_{23} - (N_{23} - N_{30})M_{20}}{(N_{20} - N_{30})D_{23}}$. **9.** $\dfrac{M_{35}}{D_{35}}$. **10.** 0.9.

SET 25

1. \$6293.82. **2.** \$3756.78. **3.** \$1064.42. **4.** \$7845.39.

SET 26

1. 23.28, 23.58, 10273.22, 595.31, 7.74, 865.66, 306.07. **2.** \$1436.65. **3.** \$18.10, \$9.12, \$4.58, \$1.53. **4.** \$4.08. **5.** \$104.76. **6.** (*a*) \$1679.15; (*b*) \$1689.15.

SET 27

2. \$42,380.62. **3.** \$39,977.20. **4.** \$22,524.40. **5.** \$33,899.72. **6.** \$39.20.

7. (*a*) $\dfrac{100N_{26} - 10S_{27} + 10S_{37}}{D_{25}}$; (*b*) $\dfrac{1000N_{30} - 200S_{31} + 200S_{35} - 200N_{61}}{D_{25}}$;

(*c*) $\dfrac{100N_{35} + 1000D_{65} + 100S_{66}}{D_{25}}$; (*d*) $\dfrac{100N_{25} + S_{26} - S_{56}}{D_{25}}$.

8. \$18.99. **10.** \$47,079.50. **11.** \$53,559.44.

SET 28

1. (*b*) $(IA)^{1}_{x:\overline{n|}} = \dfrac{R_x - R_{x+n} - nM_{x+n}}{D_x}$. **2.** \$1915.20. **3.** \$30.36.

4. \$647.89. **5.** $\dfrac{1000(M_{20} + 2M_{30} - M_{50} - 2M_{70} + D_{70})}{N_{20} - N_{40}}$.

6. $\dfrac{100(R_1 - R_{11} - 10M_{18} + 10D_{18})}{N_1 - N_{18}}$. 7. $\dfrac{1000(M_x - M_{x+30} + D_{x+30})}{N_x - N_{x+20} - R_x + R_{x+20} + 20M_{x+30}}$.

12. 1.485%.

SET 29

1. 1165.24. 2. 43,918.63. 3. 0.064. 5. \$1.06. 6. \$1.76. 7. \$1953.03. 8. \$529.56. 9. P = \$134.38, $_{10}(MV)$ = \$1126.56 (retrospective).

SET 30

1. P = \$40.3037, α_C = \$18.8881, β_C = \$41.7850, α_F = \$2.8099, β_F = \$42.8971.
Reserves:
FNL: \$38.54, \$78.06, \$118.58, \$160.13, \$202.73.
CVM: \$16.53, \$56.95, \$98.40, \$140.90, \$184.48.
FPT: 0, \$41.10, \$83.25, \$126.46, \$170.77.

2. P = \$191.1217, α_C = \$163.5313, β_C = \$198.7012, α_F = \$12.0192, β_F = \$240.3242.

Third terminal reserves: \$577.48, \$562.62, \$481.01. 3. FNL: \$4.64, \$7.29, \$7.63, \$5.34, 0. CVM and FPT: 0, \$3.72, \$5.19, \$4.09, 0. 4. \$39.42, \$118.47; \$17.71, \$98.57; \$1.40, \$83.62. 5. α_F = \$2.61 for all 11 policies. β_F: (*a*) \$14.14; (*b*) \$15.49; (*c*) \$16.41; (*d*) \$24.62; (*e*) \$45.45; (*f*) \$4.11; (*g*) \$8.40; (*h*) \$14.25; (*i*) \$17.80; (*j*) \$42.69; (*k*) \$99.44. 6. (*e*) \$21.09, \$43.10; (*j*) \$19.52, \$41.53; (*k*) \$68.99, \$91.00. Answers for the other parts are the same as for problem 5.

Note: In actual office practice, of course, α and β should be calculated to at least 4 decimals. The answers given above were actually computed to 4 decimals and then rounded to the nearest cent.

SET 31

1. α_O = \$44.9455, β_O = \$57.2147, $_1V$ = \$43.31, $_5V$ = \$280.64. 2. \$18.91, \$51.03, \$83.92, \$117.57, \$152.03. 3. \$44.13, \$100.44, \$158.23, \$217.54, \$278.41. 4. (*a*) \$2.81, \$15.08; (*b*) \$2.81, \$26.25; (*c*) \$77.75, \$90.02; (*d*) \$31.44; \$43.71; (*e*) \$2.81, \$25.71; (*f*) \$2.81, \$8.91; (*g*) \$2.81, \$3.48.

SET 32

1. α_D = \$12.4347, β_D = \$25.0411, $_1V$ = \$9.89, $_5V$ = \$105.20. 2. \$5.47, \$23.91, \$42.70, \$61.85, \$81.36. 3. \$11.16, \$33.93, \$57.22, \$81.04, \$105.40. 4. (*a*) \$4.48, \$21.25; (*b*) \$25.96, \$43.12; (*c*) \$4.68, \$21.45; (*d*) \$21.41, \$38.85; (*e*) \$14.29, \$31.44; (*f*) \$4.48, \$12.25; (*g*) \$4.48, \$6.06.

SET 33

1. α_I = \$9.1367, β_I = \$32.0336, $_1V$ = \$6.50, $_5V$ = \$131.42. 2. \$2.74, \$26.76, \$51.38, \$76.59, \$102.42. 3. \$7.82, \$37.58, \$68.06, \$99.28, \$131.26. 4. \$67.23, \$308.54, \$875.53. 5. (*a*) \$4.48, \$21.25; (*b*) \$4.48, \$32.14; (*c*) \$27.58, \$55.25; (*d*) \$4.48, \$24.83; (*e*) \$7.27, \$34.93; (*f*) \$4.48, \$28.94; (*g*) \$4.48, \$8.55.

SET 34

1. β_J = \$29.3130, $_1V$ = 0, $_5V$ = \$104.16. 2. \$105.43, \$245.75, \$759.13. 3. \$102.89, \$242.01, \$760.98. 4. (*a*) \$4.48, \$21.63; (*b*) \$4.48, \$32.14; (*c*) \$7.27, \$34.93; (*d*) \$4.48, \$25.70; (*e*) \$12.03, \$39.69; (*f*) \$4.48, \$12.37; (*g*) \$4.48, \$5.18.

SET 35

10. 10 years. 11. $P + \dfrac{P - \alpha c}{a_{x:\overline{n-1}|}}$. 12. (*a*) $40.02, $40.02; (*b*) $2.53, $42.61; (*c*) $29.57, $40.74; (*d*) $29.27, $40.76; (*e*) $19.84, $41.41; (*f*) $19.84, $41.41; (*g*) $19.84, $41.41.

SET 36

1. $223.76. 2. $330.23. 3. $585.09.

SET 37

1. $318; 7 years, 346 days. 2. $469; 11 years, 171 days. 3. $657; 5 years, $595 pure endowment.

SET 38

1. (*a*) $32.91; $81; 9 years, 31 days; (*b*) $40.27; $99; 10 years, 310 days; (*c*) $33.47; $82; 9 years, 81 days; (*d*) $65.14; $129; 16 years, 56 days; (*e*) $188.32; $464; 33 years, 165 days; (*f*) $171.01; $245; 15 years, $175 pure endowment; (*g*) $405.65.

SET 39

1. $$\frac{1015M_{[30]} + 3D_{[30]} + 2N_{[30]} - N_{50}}{0.98N_{[30]} - 0.96N_{50} - 0.02N_{[30]+3} - 0.35D_{[30]} - 0.15D_{[30]+1} - 0.1D_{[30]+2}}.$$

5. $$\frac{1015M_{[40]} + 7D_{[40]} + N_{[40]}}{0.95D_{[40]}}.$$

6. $32.92. 7. $33.99. 8. $\dfrac{M_x - M_{x+10}}{D_x} + \dfrac{D_{x+10}}{D_x} \cdot \dfrac{M'_{x+10}}{D'_{x+10}}$ where the primed symbols are at 2½% and the unprimed at 3%.

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INDEX TO SYMBOLS

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